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MATHEMATICS STAFF OF THE COLLEGE, *University of Chicago*

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Helmholtz and the nature of geometrical axioms: a segment in the history of mathematics

MORTON R. KENNER, *Southern Illinois University, Carbondale, Illinois.*

It was a hard struggle for mankind to free himself from the notion that geometry had anything to do with the real world.

"IN SHORT, MATHEMATICS is what we make it, . . . and having been made, it may at some future time even fail to be 'mathematics' any longer."[1] This simple, yet profound, observation by Professor Wilder justifies, if justification is indeed necessary, that important activity called the history of mathematics. And it also provides a clue to the use that we as teachers of mathematics can make of the history of mathematics, for the history of mathematics tells us what mathematicians made it in a given historical period and explains the influence of the presuppositions of that period.

The investigations of Hermann von Helmholtz (1821–1894) into the nature of geometrical axioms is a striking example of how the presuppositions of a historical period enter into the considerations of mathematical problems. Helmholtz typifies the radical empiricist of the nineteenth century, and his researches into the nature of geometric axioms furnish us with a picture of the empiricist's explication of geometrical axioms. As G. Stanley Hall, who was both a student and collaborator of Helmholtz before joining the faculty of Columbia University, has remarked, "The most radical side of Helmholtz's empiricism is to be found in his abstruse and largely mathematical discussions respecting the nature of the Euclidean axioms."[2] The following exposition will follow closely Helmholtz's lecture, "On the Origin and Significance of Geometrical Axioms."[3]

Helmholtz begins by asking what the origin of those propositions, which are unquestionably true yet incapable of proof, is in a science where everything else is a reasoned conclusion. He wonders if they were inherited from some divine source as the idealistic philosophers think. But yet another answer might be that the ingenuity of mathematicians has hitherto been unable to find proofs or at least explanations of these propositions. As Helmholtz rightly asserts, "The main difficulty in these inquiries [the nature and origin of geometrical axioms] is, and always has been, the readiness with which results of everyday experience become mixed up as apparent necessities of thought with the logical processes, so long as Euclid's method of constructive intuition is exclusively followed in geometry."[4]

[1] Raymond L. Wilder, *Introduction to the Foundations of Mathematics* (New York: John Wiley and Sons, 1952), p. 284.

[2] G. Stanley Hall, *Founders of Modern Psychology* (New York: D. Appleton and Co., 1912), p. 256.

[3] Hermann von Helmholtz, *Popular Lectures on Scientific Subjects* (London: Longmans, Green and Co., 1893), II, 27–73.

[4] *Ibid.*, p. 39.

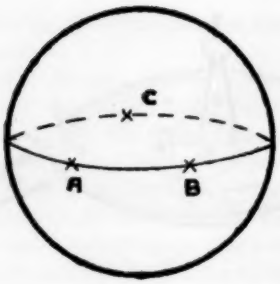


Figure 1

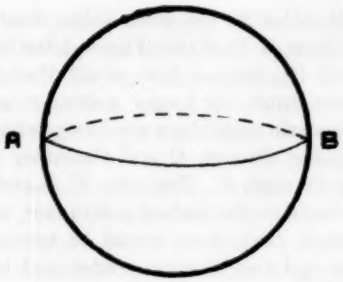


Figure 2

The foundation of much proof in Euclid, as has been established by modern research, lies in the eighth axiom of Book I of the Elements of Euclid which states: "Magnitudes which coincide with another, that is which exactly fill the same space, are equal to one another."[5] The axiom, of course, expresses the intuitive notion of congruence. But as Helmholtz rightly maintains, if we build our necesssities of thought upon this assumption, we must also assume the possibility of free translation of fixed figures with unchanged form to any part of space. And it is this latter assumption which we must examine since clearly no proof for it is given in Euclid. But let us make this latter point clear in some detail.

Imagine that we are beings of a two-dimensional space, that is, that we live and move on the surface of some solid body. Assume also that as rational beings we intend to develop a geometry for our two-dimensional universe. As in Euclidean space, a point will describe a line and a line a surface; but if our universe were only two dimensional we would be incapable of imagining a surface moving out of itself just as we, in our three-dimensional space, are incapable of imagining a solid moving out of itself. But let us get back to our two-dimensional space. Given two points in that space, we could draw a shortest line between them; however, we must be careful to recognize that the shortest lines in our space are not necessarily straight lines in the usual sense. The technical term for such lines is geodetic lines.

A geodetic line is characterized by the property that of all the paths on a given surface connecting any two points of the line, the shortest is the geodetic line itself. Clearly the straight line in ordinary Euclidean geometry is a geodetic line. We might visualize geodetics as essentially lines described by a tense (or taut) thread laid upon a surface so that it can slide freely upon that surface. Currently the term geodesic rather than geodetic is used for such lines as those described above, but in this discussion we shall use the term geodetic which follows exactly the usage of Helmholtz.[6]

To pinpoint this discussion further, let us suppose that we lived on the surface of the sphere illustrated in Figure 1. We ask what are our geodetics on this surface. As any navigator knows, they are the arcs of great circles. For example, the shortest distance from A to B is illustrated in Figure 1. It would be that arc from A to B lying along the great circle going through A and B. On the other hand, if we imagine a third point C, which is distinct from A and B, lying on the great circle through A

[5] I. Todhunter (ed.), *The Elements of Euclid* (London: J. M. Dent and Sons Ltd., 1933), p. 6.

[6] *Op. cit.*, p. 35.

and B and if we ask what is the shortest route from A to B (see Figure 1 for location of C), we see first of all that the shortest route no longer coincides with the geodetic since there are two geodetics, one going through C and the other not going through C. However, if A and B were to lie on the ends of a diameter, as in Figure 2, then there would be two geodetics and two shortest routes and they would not coincide. So we see that, at least on the surface of the sphere, it is not necessarily true that through every two points there is one and only one "shortest route."

Now suppose that we were dwellers on the surface of the sphere. We would then know nothing of parallel lines, (where by parallel lines are meant nonintersecting geodetics) for clearly, as Figure 3 illustrates, any two distinct geodetics when continued will intersect in at least two points. Furthermore, we would have no notion of similarity, for as Figure 4 illustrates, the sum of the angles of $\triangle A'B'C'$ is less than the sum of the angles of $\triangle ABC$. We would, however, have a notion of congruence, i.e., again in Figure 4,

$$\triangle A''B''C'' \cong \triangle A'B'C'.$$

Figure 3

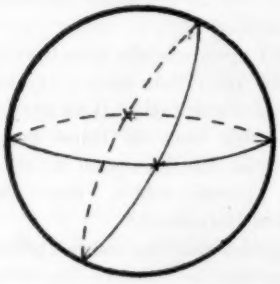
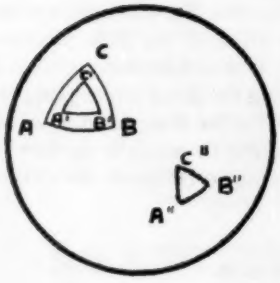


Figure 4



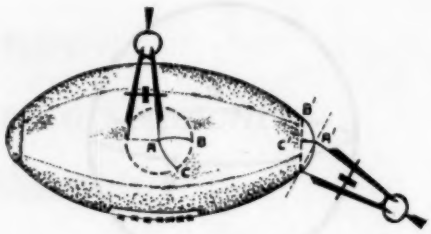


Figure 5

Thus we see that if we lived on the surface of a sphere, we would clearly set up a different system of geometric axioms from that which we are accustomed to, even though our logical powers might be exactly the same.

Now let us proceed to imagine ourselves as inhabitants of a different type of surface, say an ellipsoid, as in Figure 5. Here we could construct geodetics between any three points, but notice that triangles of equal geodetics, such as $\triangle ABC$ and $\triangle A'B'C'$, no longer necessarily have equal angles. Clearly, there are distinct triangles of equal geodetic lengths which are congruent, but there are many which are not, depending on how near the pointed end or near the blunt end of our surface we have our triangles. Thus, on such a surface we could not move figures in any way we please, i.e., preserving geodetic distances, without changing their form. Some motions would preserve form, others would not. We might also add that circles of equal radii (length measured along geodetics) would not necessarily be the same, as is illustrated in Figure 5 by the circles traced out on the surface of the ellipsoid by the two compasses. The circumference

would be greater at the blunter than at the sharper end.

Thus we see that if a surface admits of the figures lying on it being freely moved without change, the property is a special one which does not belong to every kind of surface. The condition under which a surface possesses this important property was first pointed out by Gauss. The measure of curvature, as Gauss called it, i.e., the reciprocal of the product of greatest and least curvature, must be everywhere equal over the whole extent of the surface.[7] Gauss also showed that this measure of curvature is not changed as the surface is bent without distention or contraction of any part of it. (By distention, Gauss meant stretching or distorting or tearing.) For example, a sheet could be wrapped into a cylinder and the properties dependent upon curvature would remain invariant, or stated otherwise, the geometry insofar as it depended on curvature would be the same on a cylindrical surface as it would be on a plane. Of course, there are certain niceties which must be taken care of in terms of the finiteness of the cylindrical surface but we shall pass by these difficulties at this time.

Figure 6

We must next introduce a third type of surface, the surfaces which Helmholtz called "pseudo-spheres." These essentially can be any of the three illustrated in Figure 6. The important property of these pseudo-spheres, i.e., the surfaces of these pseudo-spheres, is that again the measure of curvature is constant, only it is negative instead of positive, so that the property of congruence holds on them. That is, all figures constructed at one place can be transferred to any other with perfect continuity of form and perfect equality of all dimensions lying on the surface itself. However, on these pseudo-spheres the parallel axiom no longer is valid. The difficulty here, however, is unlike that on the surface of the sphere, where no parallels existed. Here, given a geodetic and a point not on it, we can construct an infinite number of distinct parallels through the point to the given line, i.e., geodetics which will not meet the given geodetic regardless of how far they are extended. We, of course, now only call two of these nonintersecting geodetics parallels, but here the important fact is the existence of more than one geodetic not intersecting the given geodetic.

Let us pause to sum up what we have discovered. On the surface of an ellipsoid or on a sphere, the parallel axiom does not hold since any two geodetics intersect; in an ordinary Euclidean plane there is exactly one line parallel to a given line through a given point; while on a pseudosphere given a line and a point not on the line there are infinitely many lines through the point that do not intersect the given line.

On the ellipsoid, the notion of congruence does not hold; on the sphere and on the pseudosphere the notion of congruence does hold; and in the Euclidean plane the notion of congruence holds.

On the ellipsoid, the notion of similarity does not hold. On the sphere and pseudosphere the notion of similarity does not hold. (In fact both on the sphere and pseudosphere the following theorem is true: Two triangles with three angles of one equal respectively to the three angles

[7] The notion of curvature is essentially an analytic one which involves a discussion of the Gaussian spherical representation of a surface. The least and greatest curvatures are then related to the tangents and normals of the mapping. It is outside the limitations set for this paper to discuss this concept carefully. A discussion of it, however, can be found in Hilbert and Cohn-Vossen, *Geometry and the Imagination* (New York: Chelsea Publishing Co., 1952), pp. 193–97.

of the other are congruent. Hence there are no similar noncongruent triangles.) On the Euclidean plane the notion of similarity holds.

We might tabulate these results as follows:

Surface	*Parallelism*	*Congruence*	*Similarity*
Euclidean Plane	Yes (One through point to a line)	Yes	Yes
Surface of sphere	No	Yes	No
Surface of ellipsoid	No	No	No
Surface of pseudosphere	Yes (Infinitely many)	Yes	No

It is easy analytically to extend these four surfaces to three-dimensional spaces in which the properties which we have been discussing still do or do not hold, as the case may be, in their two-dimensional analogues; but, as Helmholtz himself admits, "When we pass to space of three dimensions, we are stopped in our power of presentation [visualization] by the structure of our organs and the experiences got through them which correspond only to the space in which we live."[8]

It is possible to represent the three-dimensional analogues of the above-discussed surfaces by means of the so-called Poincare projective models, but the theory of projective models would take us far afield at the moment and certainly these models were unknown to Helmholtz. However, if the reader is interested, a thorough discussion of them can be found in the Hilbert Cohn-Vossen book, *Geometry and the Imagination*.[9]

Helmholtz maintained, nevertheless, that it was possible to imagine conditions for which the properties or measurements of Euclidean space would become what they would be either in a spherical or pseudospherical space without the use of the above-referred to Poincare models. To understand his method, let us first remind ourselves that if the dimensions of other objects in our own world were diminished or increased in like proportion at the same time, we, with our means of space perception, would be utterly unaware of the change. This would also be the case if the distortion were different in different directions. The only proviso is that our own body be changed in the same manner simultaneously and that an object in rotating assumed at each instant the amount of distortion in its different dimensions corresponding to its position at that time.

For example, let us imagine the image of the world in a convex mirror. A well-made convex mirror represents objects to all as apparently solid objects and at fixed positions behind its surface. But, as we know, the images of the distant horizon and the sun in the sky lie behind the mirror at a finite distance which, in fact, is equal to the focal length of the mirror. Between the images of the distant horizon, say, and the surface of the mirror would be found the images of all other objects before it. But, of course, those images would be diminished and flattened in proportion to their respective distances from the mirror. The image of a man measuring with a ruler a straight line from the mirror would contract more and more the farther he went, but with his shrunken rule, the man in the image would count out exactly the same number of inches as the real man, and, in general, all geometric measurements whatsoever of lines and angles taken with regular varying images of real instruments would yield exactly the same results as in the outer world. Also, all congruent bodies would coincide on being applied to one another in the mirror as in the outer world. And all lines of sight in the outer world would be represented by straight lines of sight in the mirror.

But what does this prove? It shows

[8] *Op. cit.*, p. 44.
[9] *Op. cit.*, pp. 242–63.

that we cannot tell how the man in the mirror can discover that his body is not a rigid solid. It also shows that his experience (to him) is completely interpretable in terms of Euclid's axioms. Further, if he could look out upon our world as we can look into his, he would declare our world to be a picture in a spherical mirror and would speak of us just as we speak of him. And if he could communicate with us, Helmholtz concludes that neither he nor we could convince the other that he had the true and the other the distorted world picture.

Now if instead of a convex mirror we used a surface of a sphere, and if we imagined that in the sphere moving bodies contracted as they departed from the center as the images do in a convex mirror, we would have the "mirror analogue" of a pseudospherical space. And in a manner similar to that above, Helmholtz proceeds to analyze how objects in such a pseudospherical world, were it possible to enter one, would appear to an observer whose eye measure and experiences of space had been gained, like ours, in Euclidean space. In fact, he even shows the type of lens that we might use to have this visual experience. He says:

> Now we can obtain exactly similar images of our real world, if we look through a large convex lens of corresponding negative focal length, or even through a pair of convex spectacles if ground somewhat prismatically to resemble pieces of one continuous larger lens. With these, like the convex mirror, we see remote objects as if near to us, the most remote appearing no farther distant than the focus of the lens. In going about with this lens before the eyes, we find that the objects we approach dilate exactly in the manner I have described for pseudospherical space.[10]

Helmholtz finally concludes that he has shown how we can infer from the known laws of our own perceptions the sensations which a spherical or pseudospherical world would give if it existed. What necessarily follows is, of course, that we cannot hold that the axioms of geometry in any way depend upon a priori intuition. Helmholtz sums up his own argument as follows:

> 1. The axioms of geometry, taken by themselves out of all connection with mechanical propositions, represent no relations or real things. When thus isolated, if we regard them with Kant as forms of intuition transcendentally given, they constitute a form into which any empirical content whatever will fit, and which therefore does not in any way limit or determine beforehand the nature of the content. This is true, however, not only of Euclid's axioms, but also of the axioms of spherical and pseudospherical geometry.
>
> 2. As soon as certain principles of mechanics are conjoined with the axioms of geometry, we obtain a system of propositions which has real import, and which can be verified or overturned by empirical observations, just as it can be inferred from experience. If such a system were to be taken as a transcendental form of intuition and thought, there must be assumed a preestablished harmony between form and reality.[11]

The empirical foundation of our geometric axioms is thus established by answering positively the question: Can non-Euclidean geometry become an object of intuition? For Helmholtz if the foundations of geometrical axioms were not empirical, we should be unable to make non-Euclidean geometry the object of intuition. By (1) above, this intuition is limited to no particular geometric system, and by (2) this intuition, as well as its origins, assumes real import subject to empirical observation only when conjoined with certain principles of mechanics. Were this latter not the case, we would then have to assume that our intuition of space carried along with it an a priori knowledge of empirical data, i.e., objects and their spatial relations.

In conclusion, we might point out that the development of geometry has been a steady march toward abstractness—formalism in the modern terminology. This development has progressively forced empirical subject matter into the background, and in our own times has

[10] *Op. cit.*, p. 61.

[11] *Ibid.*, p. 68.

attempted to eliminate it from geometry altogether. To regard geometry, however, as a purely deductive science is to forsake entirely the view that geometry—considered as a branch of mathematics—depends in any way upon spatial intuition. And if we consider the current abstract point of view in geometry a commonplace notion, we should remind ourselves—as a partial critique of our own preconceptions—that the slow elimination of such empirical dependence in geometrical considerations was neither easy nor always correct, and the imaginative and resourceful work of Helmholtz shows us that it was no David who finally slew the empiricists. Unless, of course, David's last name was Hilbert.

How fares the college teacher

Do you know a college teacher who is paid $18,000 for nine months of service?

There are at least 35 (by actual count) such individuals in American colleges and universities. Frequently these same individuals earn additional sums working during the summer and, in some few instances, these earnings are further supplemented by royalties and consultation fees.

There is a sharp contrast to the above picture. There are nearly 2,500 college teachers who earn less than $2,400. As many as 10,000 college teachers earn less than $4,000.

Admittedly these facts are set forth to emphasize the wide range of salaries for college instructional personnel. Averages or medians are not easy to come by in so complex a situation as college salaries. But Dr. Ray C. Maul, Assistant Director of the NEA Research Division, has prepared the nation's first "birds-eye-view" of the college professors' pay checks. Considering all ranks of teachers in all colleges and universities, the median salary during 1955–56 was $5,243.

This single figure covers many facts and factors. A closer look at the salary situation, as given us by Dr. Maul, shows that the highest individual salaries go to professors in nonstate and nonpublic universities and land-grant colleges. The median salary of professors is as high as $12,250 in one state university, while at the other extreme is a state university with a median salary of $5,100. In teachers' colleges, no salary above $10,000 has been found.

It is a spotty picture and one of sharp contrast. But on the whole, it contains some encouraging signs. Says Dr. Maul: "In addition to the many satisfactions and rewards of a nonmonetary nature, college teaching can present evidence that the top salaries are substantial. . . .

"Most of the top salaries are reported by the multiple-unit institutions, but these institutions employ the great majority of teachers. Thus, it must not be concluded that the single-unit institutions are devoid of salary attractions. Also, it must be noted that the top level salaries are found in both the public and nonpublic institutions. Noteworthy, too, is the fact that some of the most attractive salaries are paid in a region in which the general salary structure is least strong. In short, every type of institution and every region produces positive evidence that college teaching merits the consideration of our finest young people."

All this and more you'll find in Dr. Maul's excellent study, "Salaries Paid and Salary Practices in Universities, Colleges and Junior Colleges, 1955–1956." It is published as an NEA *Research Bulletin* for October, 1956.

About the time Dr. Maul released his study, an interesting sidelight on college salaries was revealed. Harvard President Nathan M. Pusey sent a report to his board of governors, in which he said, in part: "The first concern of any college should be its teachers. Harvard values excellence in her teachers and must see to it that they are correspondingly recompensed. Still, the Harvard professor is a poorer man today than he has been for generations. Despite repeated salary increases he has steadily lost ground."

—*Taken from* Edpress News Letter, *Vol.* 18, *No.* 5, *November* 12, 1956.

The readability of mathematics books

DONOVAN A. JOHNSON, *University of Minnesota, Minneapolis, Minnesota.*

The application of readability tests to mathematics textbooks yields some disconcerting results.

ONE OF OUR MOST COMMON WAYS of learning is through reading. Reading can be an easy, effective way of learning mathematics provided the material is appropriate. To be appropriate, the reading material should be accurate, interesting, readable. Thus, it would seem important that we know how to determine the readability of the mathematics materials we are using. It is the purpose of this article to describe one way of testing the readability of mathematical material and to report the results of the technique when applied to a variety of mathematics books.

As teachers we have accepted the responsibility of being reading teachers within our own field. As mathematics teachers we are aware that the reading of mathematical material is different from the reading of other subject matter material. And we are acutely aware of the difficulties our students have in reading explanations, problems, or references of a mathematical nature. With the current emphasis on the role of understanding, we need to recognize that language can be a barrier as well as an avenue to understanding. Reading becomes a barrier when the words used fail to convey meaning to the reader. The reasons for such failure can be traced to a variety of causes. Difficulties in reading mathematics material may come from a general reading inadequacy, from the use of technical language, from the use of symbols, formulas, and graphs with which the reader is unfamiliar, or from too concise language. How readable are the materials we use in our mathematics classes?

An extensive study of the reading problem in mathematics was made by Eagle.[1] He used a variety of tests to find the relationship of certain reading abilities to success in ninth grade mathematics. Eagle found that:

(1) General reading comprehension is more highly related to mental age than to the reading of mathematical material.

(2) General vocabulary is less related to successful achievement in mathematics than is mathematics vocabulary.

(3) Reading speed is a significant, but variable, factor in mathematics achievement. For low ability students, slow readers tended to have low mathematics scores. For average ability students, slow readers tended to have high mathematics scores. But for high ability students, fast readers tended to have high mathematics achievement.

(4) Interpreting formulas and graphs was more highly correlated with success in mathematics than general vocabulary, reading comprehension, or mental age.

In a somewhat similar study, Pitts[2] sought to determine the relationship between reading level, mental ability, and mathematical competence as measured by the Davis test of Mathematical Competence. Pitts concluded that the scores on

[1] Edwin Eagle, "The Relationship of Certain Reading Abilities to Success in Mathematics," THE MATHEMATICS TEACHER, XLI (April 1948), 175–79.

[2] Raymond C. Pitts, "Relationship Between Functional Competence in Mathematics and Reading Grade Levels, Mental Ability and Age," *Journal of Educational Psychology*, XLIII (December 1952), 486–92.

the Davis test were significantly related to reading level score, but were not related to I.Q. scores. From the studies summarized above, it is apparent that the reading of mathematical material is a specific skill. This specialized skill in reading mathematics material is more significantly related to mathematics achievement than to mental ability or general reading skill.

Many methods of obtaining a rating of readability have been proposed by educators. A survey of the different methods indicates that the Flesch and Dale-Chall methods are used most frequently and with the best results. These formulas are easily applied and give reliable, valid measures. The scores obtained are highly related to students' comprehension and speed in reading articles. The Dale-Chall formula measures the readability of material mainly by counting the words outside a vocabulary list of three thousand words. The basic element in reading difficulty, according to this formula, is vocabulary difficulty. In view of the specialized vocabulary used in mathematics, the Dale-Chall formula was not used in this study.

The Flesch formulas[3] may be used to measure "reading ease" and "human interest." The measure of reading ease is based on sentence length and word length for samples of the reading material. Sentence length is determined by finding the average number of words per sentence. Word length is based on the average number of one-syllable words per hundred words. Reading ease is then computed by the formula: $R.E. = 1.599 \times$ *average number of one-syllable words* $- 1.015 \times$ *average number of words per sentence* $- 31.517$. This is a modification of Flesch's original formula. In this study, the constants were rounded off to 1.6, 1.0, and 31.5 respectively.

The measure of human interest is based on personal words and personal sentences. Personal words include first, second, and third person pronouns and words of natural masculine or feminine gender. Personal sentences include spoken sentences, speech tags, questions, commands, and exclamations. Human interest is then computed with the formula: $H.I. = 3.635 \times$ *personal words* $+ 3.14 \times$ *personal sentences*. In view of the nature of most mathematics books, it is apparent that it is not appropriate to use this human interest formula on mathematics books.

These formulas produce scores ranging from 0 to 100. A score of 100 corresponds to the prediction that a child who has completed the fourth grade will be able to answer 75 per cent of the questions based on the passage to which the formula was applied. Table 1 relates the reading ease score to approximate grade level and a description of the style.

TABLE 1—READING EASE SCORES RELATED TO GRADE LEVEL[4]

Reading Easy Score	*Grade Level*	*Description of Style*	*Per Cent of U.S. Adults Attaining Grade Level**
94–90	5.5	Very easy	92%
89–87	6.0		
86–82	6.5	Easy	91%
81–77	7.0		
76–72	7.5	Fairly easy	88%
71–67	8.0		
66–62	9.0	Standard	83%
61–60	9.5		
59–58	10.0		
57–52	11.5	Fairly difficult	54%
51–47	13.0		
46–42	14.0	Difficult, college	33%
41–35	15.0		

* 1950 census.

The Flesch formula has had surprisingly wide use. It has been applied to the meas-

[3] Rudolf Flesch, *How to Test Readability* (New York: Harper and Brothers, 1951).

[4] Rudolf Flesch, *op. cit.*

urement of readability of corporate reports, house papers, farm papers, *Time* and *Newsweek*, and public opinion polls. Johnson and Bond[5] used the formula to determine the readability of vocational tests used in counseling, such as the Bell Adjustment Inventory (grade 7.0), the Kuder Occupational Preference Record (grade 9.5), and the Strong Vocational Interest Blank (grade 15.0). The Lorge formula was applied to the Minnesota Multi-Phasic Inventory as a check with the Flesch formula and very similar results were obtained (6.0 and 6.1). However, the Dall-Chall formula rated the Terman-McNemar Information test at 12.0 compared to a Flesch rating of 10.0. Extensive tests of the readability of science texts using the Flesch formula have been made by Mallinson and his associates.[6]

A comparison of the ratings which various formulas gave for the readability of adult materials was made by Klare.[7] He used five different formulas to rate fifty-two books and compared the ratings. In terms of rating by grade level, the highest grade level rating that a given book would receive on the average was given by the Washburne-Morphett formula, next highest by the Flesch, then Dale-Chall, next Lewerenz, and lowest by Gray-Leary. Klare concludes that the Flesch, Gray-Leary, and Dale-Chall gave the best measures of readability and that there is little difference between the readability measures obtained by these three Gray-Leary, Flesch, or Dale-Chall formulas.

For mathematics books, Flesch suggests that arithmetic portions be omitted. Faison,[8] however, developed a variation in his test of readability of children's books. In using the Flesch formula he counted each individual digit of large numbers as a word. He also considered each practice exercise as a sentence since it represented a complete unit. Using this modification, he found seventh grade mathematics texts to have reading ease scores from 55 to 58 (grade 11.1 to 10.5). For eighth grade texts the scores range from 58 to 62. Yet of the five subject texts examined, mathematics books did not rank as the most difficult.

This study used the modified Flesch formula to measure the readability of a variety of mathematics books. This method involves finding the average number of words per sentence, the average number of one-syllable words, and converts these figures into a readability score by means of the formula previously described. Twenty-five samples of one hundred words were chosen from each book by selecting page numbers using Fisher and Yates' tables of random numbers.

The samples selected include only sentence material. This means that non-sentence material such as pure computations, equation solving, geometric proof, titles of chapters, and illustrative problems are not part of the content examined. If material of the non-sentence type were included in the data, the reading ease scores could not be interpreted in terms of the standards based on sentence material and sentence length.

All samples counted start at the beginning of a paragraph on the page selected and include one hundred words. The word count includes all words, numbers, symbols, letters, and abbreviations in the sample passage. All groups of letters, numbers, or symbols which are surrounded by a white space are counted as one word. All abbreviations are treated as whole words, as are mathematical symbols encountered in sentences, e.g., m.p.h. counts as one word with three syllables; k.w.h. as one word with four syllables, ‖ (parallel) as one word with three syllables. When

[5] Ralph H. Johnson and Guy L. Bond, "Reading Ease of Commonly Used Tests," *Journal of Applied Psychology*, XXXIV (October 1950), 319–24.

[6] George G. Mallinson, "The Readability of High School Science Texts," *The Science Teacher*, XVIII (November 1951), 253–56.

[7] George R. Klare, "Measures of Readability of Written Communication: An Evaluation," *Journal of Educational Psychology*, XLIII (1952), 385–99.

[8] Miles A. Tinker, "Readability of Mathematical Tables," *Journal of Applied Psychology*, XXXVIII (June 1954), 436–42.

letters are used to denote geometric forms, the complete figure designation is counted as one word, but each letter as a syllable, e.g., *ABC* representing a triangle is counted as one word with three syllables. Letters and numbers separated by visible signs of operation or equality are treated as separate words. All signs of operation are treated as mathematical symbols and thus each sign as one word. Sentences are counted according to the instructions "count as a sentence each unit of thought that is independent of another sentence or clause, if its end is marked by a period, question mark, exclamation point, semicolon or colon."[9] If the 100-word mark does not fall at the end of a sentence, the part is counted as a full sentence if the mark falls after the halfway point.

In each sample, the number of one-syllable words was counted as they are pronounced. Numbers, symbols, and abbreviations are counted according to the pronunciation of the word they represent.

SUMMARY OF STEPS IN FINDING A READING EASE SCORE

1. Select a random sample of written material. Starting with a sentence, count off 100 words.
2. Count the number of sentences in this sample.
3. Count the number of one-syllable words in the sample.
4. Compute the Reading Ease (*R.E.*) of the sample by substituting in the formula:

$R.E. = 1.6s - 1.0w - 31.5$

s = number of one-syllable words

w = average number of words per sentence

The mathematics books tested included three seventh grade arithmetic texts, three eighth grade arithmetic texts, two elementary algebra texts, three plane geometry texts, and seven supplementary mathematics books. The data obtained are summarized in Table 2.

The data reported in Table 2 above show that the seventh and eighth grade arithmetic texts analyzed generally require a high level of reading skill. The algebra books were of approximately the right grade level. It is surprising to find that the geometry texts seem to have readability scores at or below the grade for which they are written.

TABLE 2—READABILITY OF MATHEMATICS BOOKS

Books		*Reading Ease Score*	*Grade Level*
Seventh Grade Textbooks	I	72	8.0
	II	76	7.5
	III	65	9.0
Eighth Grade Textbooks	I	71	8.0
	II	63	9.0
	III	65	9.0
Algebra Textbooks	I	61	9.5
	II	62	9.0
Geometry Textbooks	I	63	9.0
	II	63	9.0
	III	61	9.5
Supplementary books	I	67	8.0
	II	53	11.5
	III	68	8.0
	IV	71	8.0
	V	40	15.0
	VI	48	13.0
	VII	54	11.5

The reading difficulty of the supplementary books which were examined varies greatly, as one would expect from the books included. The supplementary books tested included the following: *Mathematics at the Fireside, Mathematics in Western Culture, How Much and How Many, Flatland, The Universe and Dr. Einstein, Makers of Mathematics, Mathematics and the Imagination.* The reading ease scores given these books appears to verify further the reliability of the Flesch formula in measuring readability. The variation in reading difficulty of these supplementary books indicates that the assignment of readings in them to high school students needs to be made with considerable care.

[9] Rudolf Flesch, *op. cit.*

Thompson[10] made a similar analysis of ninth grade mathematics textbooks using the Flesch nomograph, based on the formula: $R.E. = 1.015L + .846\ S - 206.835$. In this formula, L represents the average sentence length and S the number of syllables per hundred words. Thompson made adaptations of word counts to fit mathematical material such as the following.

When a formula was encountered in a sampling of reading material, each literal term was counted by Thompson as one word and each letter or factor as one syllable. Numerators and denominators were surveyed separately for syllables, but the entire fraction was counted as one word. Exponents were counted as separate words with the same syllable count as the power. A quantity inside a parenthesis was taken as one word; the parenthesis counted as one syllable besides counting each sign and factor inside the parenthesis. In dealing with whole numbers, each family of three digits was counted as one syllable, e.g., 5,604,000 has only three syllables. For decimal values, the decimal point is counted as one syllable and each family of three places as one syllable, e.g., .5 has two syllables and 3,456.8 has a count of four. The results obtained by Thompson are tabulated in Table 3.

TABLE 3—Readability of Ninth Grade Mathematics Textbooks

Books		*Reading Ease Score*	*Grade Level*
General Mathematics Textbooks	I	67	8.0
	II	62	9.0
	III	66	9.0
	IV	76	7.5
Integrated Mathematics	I	60	9.5
	II	62	9.5

[10] Roger Thompson, "Readability of Ninth Grade Mathematics Textbooks," (Unpublished Master's Paper, University of Minnesota, 1955).

The data in both Tables 2 and 3 indicate that the level of reading difficulty of textbooks for grades 7, 8, and 9 vary greatly. The variation for seventh grade texts (7.5 to 9.0) is as much and the same as for general mathematics texts. On the other hand, the reading level of algebra, geometry, and integrated mathematics textbooks which were examined is very uniform. Whether or not a mathematics textbook for a given grade should have a reading level equivalent to the grade for which it is written or a lower grade is another question. In view of the individual differences found in the typical class, it is likely that a general mathematics text should have a readability score at least one grade below that at which it is used.

A further analysis of the reading samples of ninth grade texts was made by Thompson to determine whether different types of expository material varied in readability. Explanatory material included all sample passages written to build understanding, e.g., how to follow rules, how to solve an equation, the meaning of words, etc. Enrichment material included all sample passages where the purpose is supplementing information, e.g., historical incidents, recreations, and applications. Problem material included all word problems that require the student to find a solution, to compare or discuss plans and procedures, to prove a solution or procedure to be true, and test items. The results are summarized in Table 4.

From Table 4 it is apparent that problem material is consistently easier reading material than explanatory or enrichment material. This is as it should be in view of the difficulties students have in perceiving relationships in stated problems. Again there is great variation in the readability scores from one text to another, as well as within any given text.

The results obtained in this study would seem to indicate that the Flesch formula is suitable for measuring the readability of mathematics books. In view of the varia-

TABLE 4—READABILITY OF DIFFERENT TYPES OF MATERIAL

	Median "Reading Ease" Scores						
	General Mathematics Texts			*Integrated Mathematics Texts*			*All Books*
	I	II	III	IV	V	VI	
Explanatory Material	8.0	10.5	10.0	7.5	9.0	10.0	9.0
Informative Material	10.0	10.0	10.5	6.5	11.0	10.5	10.0
Explanatory and Informative Together	9.0	10.0	10.0	7.5	10.0	10.0	9.5
Problem Material	8.0	8.5	7.5	7.0	8.0	8.5	8.0

tion in readability of different texts for a given grade, readability scores should be used in the selection of a text. It is possible to select samples, make word, syllable, and sentence counts, and compute the readability of a text in a few hours. Likewise the authors and publishers of mathematics textbooks should use readability tests to assure textbooks of the proper level of reading difficulty. The readability level that is most appropriate will vary depending on the type of material and the type of student most likely to use the book.

Mathematics in the world's schools

"The present shortage of scientists is due not to the failure of the schools, but to their tremendous success," asserts John Mayor, director of the science-teaching improvement program of the American Association for the Advancement of Science.

Today's great demand for scientific and technical skill, Dr. Mayor told a Texas audience, "is due to the fact that America's schools already have done such a good job of training scientists who have made possible advances that call for more technicians." He said that the future of America's progress and development depends on adequate science and mathematics programs and teachers in secondary schools. He applauded current trends in the teaching of mathematics and explained how new curriculums permit the best students to learn more in high schools.

In all the countries (65 in all) mathematics is to the forefront in secondary education. Its importance is revealed by an analysis of its status as a required or nonrequired subject, above all in the science sections.

The details of that status are given below.

(a) In two thirds of the countries mathematics is a required subject in all years and in all sections and classes of general secondary education.

(b) In all the countries except two (Philippines and Vietnam, where mathematical instruction begins only in the second year) mathematics is a required subject throughout the lower secondary cycle.

(c) A required subject in the lower secondary cycle, mathematics in some ten of the countries becomes an optional subject in the upper cycle, or is dropped altogether in the last year or years of certain sections.

(d) In some twelve of the countries mathematics remains a required subject in the upper secondary cycle only for those pupils taking their school-leaving certificate in science.

(e) Differentiation on the basis of sex, in respect of whether mathematics is a required or nonrequired subject, is relatively rare. In Denmark, in the *realklasse*, all the mathematical branches taken are optional for girls, but for the boys this is not the case, while in India arithmetic, algebra, and geometry are required for boys, but only the first of these three branches for girls.

(f) Optional, additional courses in mathematics are offered in the upper secondary cycle in a few of the countries (France, Laos, Monaco, Nepal, Pakistan). Pupils with aptitude and an interest in science are thus enabled to do work at a higher level than is required by their curriculum. In the U.S.S.R. and other countries in Eastern Europe, moreover, such pupils can optionally pursue their mathematical studies outside class hours, in the school mathematics societies.—*Taken from* Teaching of Mathematics in Secondary Schools, XIXth International Conference on Public Education, *Geneva*, 1956, *Publication* 172.

The wolf at the door—1957 style

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The facts of installment buying and repossession laws are explored in a consumer mathematics course.

CONSUMER PROBLEMS have long been regarded as an essential part of instruction in general mathematics. Problems concerning economical purchasing, the use of credit and "easy" payments, the nature and use of various kinds of insurance, and gearing income to expenditures, are to be found in most elementary texts in arithmetic and high school general mathematics. Indeed, the subject matter is so sufficiently detailed that many high schools have established courses in consumer problems or family economics. These are sometimes taught by mathematics teachers and often taught by social studies teachers, business education or home economics teachers.

The author has experimented in recent years at the ninth grade level with problems geared more to the consequences of unfavorable decisions than to arriving at favorable ones. A great deal of student interest can be generated, for instance, by discussing the general question, "What happens when you don't meet your payments?" Most students will answer, "They take what you have been buying away." The teacher then asks, "Is that the end of the story? Do you still owe any money?" It is at this point that a teacher having adequate knowledge of the repossession laws in his state and how they are designed to protect both buyer and seller, can illuminate vast areas of student ignorance. Much surprise and some indignation ("They can't do that!") will generally result. Students then begin to volunteer family experiences, and the resulting discussion may well last several class periods.

The problems which follows are typical of those which arouse discussion. The arithmetic is simple, but the problems illustrate the legal principles in unfolding complexity.

1. A man failed to meet the payments on his car, and it was repossessed by the finance company. He owed \$583.25 on the car at the time. It was sold at auction for \$450. Did he have to make the rest of the payments? If not, what happened? (Answer: In most states, the law requires that repossessed goods be sold at public auction, and the sale price be credited to the outstanding debt. The man still owed \$133.25. If he failed to pay, the finance company could take him to court and have his wages garnisheed for that amount, plus court costs and interest charges.)

2. Suppose the car in Problem 1 were auctioned for \$650. What would happen then? (Answer: Since the debt outstanding was less than \$650, the buyer is entitled to a refund of the difference, in this case \$66.75. In most cases, this money is not paid because the buyer is ignorant of his legal rights. Even if he knows his legal rights, he will have difficulty collecting from an unscrupulous company, since he is usually not in a position to take them to court. It is well to note, at this point, that the public auction, in theory, is designed to protect buyer and seller alike, since the auction price is the "true" market value. In practice, the auctions are often "rigged" in favor of used car dealers. The foregoing is not meant to give the impression that all finance companies and retail stores are guilty of shady practices. Many are honest businesses, but we must not close our

eyes to the facts of life.)

3. A family owed \$135.48 on a television set. When they did not pay, it was repossessed and sold at auction for \$90. What happened? (Answer: As in the first case, they still owed \$45.48.)

4. Suppose the family in Problem 3 decided to sell the television set before missing a payment, and sold it for \$150 through a classified ad. How much would they have saved by comparison? (Answer: Most students will answer that they sold the set for \$14.52 more than they owed, said amount constituting the saving. The difference between the \$90 auction price and the \$150 sale price indicates an actual difference of \$60. This problem is designed to indicate an alternate and better solution to a distressing financial problem. Such a solution is not always possible, since the sale contract may specify that the buyer cannot resell the article without the permission of the mortgagor until it is completely paid for. Students can be informed, at this point, that most people sign these contracts without reading them.)

5. A family lost their home because they could not meet the mortgage payments. The mortgage balance was \$6722.85. The house was sold for \$8000. What should happen? (The laws regarding the credit sale of other goods generally apply to housing, with technical variations. The family is entitled to a refund of \$1277.15, less charges.)

6. A man lost his television set by repossession. He owed \$119.28 on it at the time. The set was auctioned for \$125. There was a repossession charge of \$15 and an auctioneer's fee of \$12. What was the result? (Answer: The buyer was responsible for the charges as well as the original contract. He still owed \$21.28.)

7. A family owing a mortgage of \$5,812.80 lost their home for failure to meet the payments. By the time it was sold for \$7200, the family had accumulated interest debts of \$412.50 and other charges of \$115.80. Did they have anything left, and how much? (Answer: They were entitled to \$858.90 after all charges were paid. The amount of interest seems high, but it takes about eighteen months to foreclose a mortgage. As stated previously, many people do not get refunds because they are ignorant of their legal rights.)

8. A man owed \$18 on a suit and did not pay. He reasoned that the suit was practically worn out and it was cheaper to let the store take it back. The company collector called at his house, refused the suit, and had the company institute a claim at the small claims court. The company claimed a collecting charge of \$2, and the court charges were \$10.50. What happened? (Small claims procedure varies from state to state, but the most likely occurrence would be attachment of the man's wages for \$30.50. Nondurable or "soft" goods are generally not repossessed. A problem of this type is a good opportunity to show that the important factor is not the article itself, but the signed contract.)

9. Joe Bell owed \$592.50 on a car, and could not meet his payments. He put a classified ad in the paper to sell the car, but the best offer he got was \$550.00. Reasoning that it was not worth selling, since he owed more than \$550, he allowed the finance company to take it. It was auctioned for \$375. The repossession charge was \$18 and the auctioneer's charge was \$25. What did he have to pay? What did his faulty reasoning cost him? (Answer: After the charges were added to the balance, and \$375 deducted, he owed \$260.50. Selling for \$550 would have left him with a debt of \$42.50, a difference of \$218, the price of faulty reasoning or ignorance.)

10. What is an open charge payment account and how does it work? (Some unscrupulous stores can be found in most large cities, which offer contracts by which people can charge new articles on monthly payments without completing, up to that date, the payments on an article still under debt. This arrangement has several as-

pects. One is that the buyer never makes a down payment, undoubtedly a great attraction. Another is that the buyer hardly ever knows how many payments he has left, or what his balance is. Although he may find these out by asking, one thing he never knows is the rate of interest, which is often as high as 40%. While this may seem shocking to the reader, it is nevertheless true that in many states which place maximum rates on interest, there is no maximum rate on carrying charges, the same thing under a different name. In other states, the maximum rate on such carrying charges is more liberal than for interest on bank loans. The most shocking aspect of the whole business is a clause in the contract, generally signed without reading, which entitles the seller to repossess, not only the article under charge, but *every article ever purchased on the account*, if a payment is missed. Fortunately, there are many more legitimate businesses than stores of this type, and the consumer can avoid them.)

The author has not, as yet, attempted to evaluate in a scientific way, the end effect, in a permanent sense, of class discussions centered about the subject of repossessed goods. There is good reason to believe, on the basis of student comment, and some test questions, that a great deal of important learning took place. The subject matter may be treated in anywhere from two to ten class periods, at the discretion of the teacher, who is in a position to judge when interest is waning. For those willing to dig into our current consumer picture, it may be possible to strike educational oil.

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"Definition I. A variable *is a symbol which represents a number that may vary within a given range.*

"Definition II. A constant *is a symbol which represents a number that does not change in any discussion or situation.*

"Note 1: The values of some symbols, such as π, 1, 2, 3, . . . never change, and these are called *absolute constants.*

"Note 2: We frequently use a letter whose value is temporarily unknown, but which is fixed by the conditions of the problem. For example, in the equation

$$2x-1=4x-3$$

the value of x is unknown until the solution reveals that its value is 1. Hence, in this problem, x is not a variable, but is a constant whose value is to be determined."

Editor's Note: This quotation is a sample of the terminology all too commonly found in some texts designed for college use. Is the careful definition of terms evident in this sample? (The italicized portions are as found in the book.) It is interesting to conjecture whether, according to the above, x is a variable in a quadratic equation. In an equation of 10th degree? Of 50th degree? Also, what is "a number that may vary"?

Equations and geometric loci; a logical synthesis[1]

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Teachers of elementary mathematics are beginning to realize that certain elementary ideas are fundamental to the study of mathematics. This article presents some of these ideas and applies them to topics studied in secondary mathematics.

THE SOLUTION of equations and the finding of geometric loci indicate that there is a common logical foundation of concepts relating sets and propositions.

The purpose of this paper is to present, in an elementary manner, this underlying structure and to show by illustrations how it can be used to advantage in secondary education.

I. STATEMENTS, PROPERTIES, RELATIONS, AND SENTENCES

A. Let us consider the three phrases:

$\triangle ABC$ is isosceles.

7 is a multiple of 2.

$3+8=11$

These are logical statements; that is, they are sentences which can be judged as true or false. The equality $3+8=11$ is true; but the statement 7 is a multiple of 2 is false. As to "$\triangle ABC$ is isosceles,"—it is true or false, depending on the triangle designated by ABC.

The three statements are, furthermore, singular because they concern only given individual elements.

B. In place of the given elements, let us substitute in these sentences an undetermined element, designated for instance by the letter x. Then we obtain the sentences:

$\triangle x$ is isosceles.

(1) x is a multiple of 2.

$x+8=11$

We can also introduce several undetermined arguments x, y, z:

x is a multiple of y.

(2) $x+y=1$

$x+y=z$

The expressions we obtain in this way are no longer true or false sentences. They are no longer logical statements, but forms which will give such statements if we replace x, y, and z by individual specified elements. Modern logicians call such expressions *open sentences* of which the letters x, y, z represent the *terms* or *arguments*.

C. The open sentences (1) which contain only one argument, express a property of that argument:

being isosceles
being a multiple of 2
being the difference between 11 and 8

We will designate the properties by P and Q, and the corresponding open sentences by $P(x)$ and $Q(x)$.

The sentences (2) which involve several

[1] Translated by Julius H. Hlavaty, DeWitt Clinton High School, New York, and Howard F. Fehr, Teachers College, Columbia University, New York.

arguments express a relation between these arguments:

being a multiple of
having a sum equal to 11
having a sum equal to z

We shall designate the relations by R, S, . . . and the corresponding open sentences by $R(x, y)$, $S(x, y, z)$, ...

If we consider a relation between x and y as a property of the ordered pair (x, y) we can use the concept of property in all cases.

Frequently, moreover, a property of a thing is a relation between parts of that thing.

For example, the property "a quadrilateral is a parallelogram" is equivalent to the relation "the opposite sides of a quadrilateral are parallel."

A property or a relation may also be designated by the word "condition."

We may then conclude: An open sentence expresses a condition bearing on one or more undetermined arguments.

We shall designate an open sentence by the notation $P(x, \ldots)$.

II. Domain

When we consider a sentence, a question arises immediately: by what elements can we eventually replace the indefinite argument to obtain a true statement? Or in other words, what are the elements which satisfy the property or condition?

Before answering this question, it is necessary at the very start, in order that the problem be clearly stated, to know from what set it is permissible to select individual elements to substitute for the arguments of the sentence so that the resulting statements may have meaning, that is to say, that they be either true or false.

The set of elements from which we can select values for the argument of a sentence must be given. This set we will call the set of reference or the *domain*.

The sentences may themselves suggest the domain.

In "Δx is isosceles" we are thinking of triangles, but it should be further specified whether they are plane triangles or spherical triangles.

For "x is a multiple of 2" we know that it is a question of whole numbers, but are they natural numbers? Should we include 0 and the negative whole numbers?

As for the equation "$x+y=11$," should we take values of x and y that are whole numbers, or real numbers, or complex numbers?

These are niceties on which it might be superfluous to insist, if they were always present in the consciousness of pupils and mathematicians. But we find even from the pen of M. Emile Borel, in the preface to his *Algebra and Geometry of the Second Degree*, the following opinion:

> In the course of this study we introduce the fundamental notion of imaginary roots. It is actually a serious pedagogic error to teach, as some official syllabi prescribe, that equations of the second degree whose roots are imaginary do not have roots. For in this way we assume as not having happened one of the most beautiful and fruitful discoveries of the human mind; pupils are habituated to repeat inexact statements which they will have to learn to contradict later.

Certainly it is committing an error of logic to assert, without qualification, that an equation of the second degree whose discriminant is negative has no roots, because the nature of the roots under discussion has not been made clear. But how should we express ourselves in a first presentation? Simply by making explicit the domain: an equation of the second degree whose coefficients are rational numbers (or real numbers) and whose discriminant is negative has no root which is a positive number, or zero, or a negative number. Such a statement is clear and it does not run the risk of being contradicted by the introduction of imaginary numbers.

After having defined parallel lines as lines in the same plane which have no points in common, we later come to consider parallel lines as meeting at infinity. It should be made clear that in going from the first statement to the second, we have

added to the plane of points at finite distances—the only ones considered until then—certain ideal elements or points.

In analytic geometry it is important to specify, when it becomes necessary, whether the co-ordinate axes, the points, and the coefficients under discussion are real or imaginary.

In the same sphere of ideas, to say that the quadrature of the circle is impossible has no meaning, unless we specify the nature of the lines which it is permissible to use in the construction.

If we seem to press this point too much, it is because we run the risk of being obscure unless we are sufficiently explicit with students.

Let us return to M. Borel's preface. The author continues with conviction:

> It should be thoroughly understood that imaginary numbers exist to the same degree as do all the other mathematical concepts that have been imagined by mathematicians. The circle, the sphere, the ellipse, the hyperbola, the sine, the cosine, logarithms, etc. exist because mathematicians have created them. It is not necessary, of course, to study them in all classes, but one should never deny their existence. . . .

Obviously, few adult mathematicians will today deny the existence of imaginary numbers. But history is here to remind us how much time has been needed, how much resistance the inventors had to overcome, before others recognized the existence of zero, of irrational numbers, of negative numbers, of imaginary numbers, of ideal points and imaginary points—to mention only the most well-known instances.

This slow historic gestation has doubtless served to demote mathematical beings and relations from the Olympus of Platonic verities to the point where they came to be considered as marvelous products of human inventiveness. But of this millennial creation we tend to present our pupils only the end products. Do we help them sufficiently to elaborate their own mathematics? Do we suggest to them situations which encourage them too to create abstractions? Are we not, most often, satisfied with transmitting in a dogmatic fashion mathematics all made? Think of the successive extensions of the number system where instead of demonstrating genetically the successive amplifications of the number domain, we limit ourselves to a formal statement of the definitions of new numbers. How many creative moments are thus let slip, only to be replaced by decrees which offer but dry dust to young intelligences?

When we wish to elaborate a concept by starting from the concrete, do we always do it felicitously? You know these phrases (we copy them from a textbook on geometry): "The simplest of all lines is the straight line. A taut string gives us an image of it."

Again, further on, "Hereafter, when we speak of lines, we will mean unlimited lines, although in reality they will be limited by the size of the page."

What an equivocation—in which are confounded concrete physical entities perceived through our experience and abstract concepts conceived by our imagination. What can exist in an adolescent mind when we think so little of what we suggest to it? But, you will say, we talk about these things so rarely, and besides they do not appear on examinations. Moreover, these ideas become more precise later on.

All the same we must be concerned with what our pupils think when we use the words "line," "segment," "circle," and "space." Question and see how many pupils know that on a segment there are points other than the endpoints, that an abstract circle has an infinity of points, that the space you speak of during the course is something different from the interior of the classroom. Question and observe, and you will be amazed at the variety of answers. You will then see the necessity of helping our pupils, by all possible means, to create the mathematics that they have to know.

Here we are, you will say, a long way from geometric loci and equations. On the contrary, we are at the very foundation of

our subject, because we are talking about what our pupils know of number, of points, of lines, and of the relations connecting them.

III. The extension of a property: existence, uniqueness, identity

A. Once we have specified the domain of reference which contains the elements which we can substitute for the arguments of a sentence $P(x, \cdots)$ to give it meaning, the question arises whether there is at least one element in this domain, say x_1, for which the statement $P(x_1, \cdots)$ is true. If there is, x_1 is said to satisfy or verify the sentence $P(x, \cdots)$ or the property P, or that x_1 is a solution of $P(x, \cdots)$. In such a case, the sentence "There exists at least one x such that $P(x)$" is true.

Such a theorem is called an existence theorem. We symbolize it by $(E_x)P(x, \cdots)$, "there exists at least one x such that $P(x)$."

B. We can also consider the set having for its elements all x's satisfying $P(x, \cdots)$. Therefore, with every property P, or sentence $P(x, \cdots)$ we associate the set $S(P)$ of elements which satisfy it within the given domain.

The set $S(P)$ associated with the property P is called the *extension* of the property P, while all properties, such as P, which are satisfied by all the elements of the set $S(P)$ belong to the *intension* of that set.

With the set $S(P)$ is associated the concept of *variable* or *generating element* which satisfies $P(x, \cdots)$, that is to say, any element of $S(P)$. This is a concept that we must take great care that our pupils acquire because they think, at the beginning particularly, only of specific, individual instances.

When $S(P)$ contains only a finite number of elements, we shall say that the solutions of $P(x, \cdots)$ are determined.

In particular, if $S(P)$ contains only one element, we shall say that the solution of $P(x, \cdots)$ exists and is unique.

Finally, if $P(x, \cdots)$ has no solution at all in the domain, $S(P)$ shall be the *null set* containing no elements, and the property P shall be said to be impossible. Is it necessary to emphasize that the set $S(P)$ depends on the domain of reference from which the elements are selected which substituted in $P(x)$ give it meaning? For example, on the sphere there are birectangular and trirectangular triangles, while in the plane they do not exist.

In the development of the mathematical thinking of our pupils, as in the historical evolution of mathematics, there are successive expansions of domains. Thus, a condition impossible in one domain may be satisfied in an extended domain.

As we have already suggested in connection with the solution of equations, if the same condition is being considered in two domains, we should, to avoid errors, state explicitly the domain, which is ordinarily taken for granted.

For example, if we consider the locus of points equidistant from the extremities of a line segment, first in the plane, then in space, it will be useful to say, successively: "In the plane, the locus of points equidistant from the ends of a line segment is the perpendicular bisector of the segment. In space, the locus of points . . . , is the plane which is perpendicular to the line segment at its midpoint."

C. Usually, the set $S(P)$ of elements which satisfy a sentence $P(x)$ is only a part of the domain.

A notable case is that in which $P(x)$ is satisfied by every element of the domain; in such a case, the set $S(P)$ is the domain itself. We shall then say that $P(x)$ is an identity. We shall write $(A_x)P(x)$, "for all x,$P(x)$."

A proposition so obtained is called a *universal* because it applies to all elements of the domain.

For example, if we represent by "$x=y$" the statement "x and y designate the same element" we shall have, in all domains, the identity "$x=x$," it being understood that the same letter will designate the same element whenever it appears in a given

relation.

We know that for all numbers considered in elementary algebra, the sentence $(x+y)^2=x^2+2xy+y^2$ is satisfied. It is therefore an identity in that domain.

D. The notions which we have introduced enable us to make clear the logical nature of the three types of equalities met with in elementary algebra.

1. A numerical equality, $3+5=8$, is a singular statement.

2. The literal identity $x+y=y+x$ gives a universal statement always true:

$$(A_x, A_y)(x+y=y+x).$$

3. An equation $x+y=z$ is an open sentence; it is neither true or false as long as x, y, z are not specified.

IV. Implication of properties; inclusion of sets

The most frequent structure of a theorem is *implication* between properties or sentences.

We know that, if in every case in which a property P is satisfied, the property Q is also necessarily satisfied, we say that P implies Q.

We write $P \rightarrow Q$, "if P, then Q"

To establish the truth of such an implication, it is necessary to show that if any x whatever satisfies $P(x)$, it also necessarily satisfies $Q(x)$.

If the implication $P \rightarrow Q$ is true, it is *sufficient* that the property P be satisfied in order that the property Q be *necessarily* true. So that the property P is a *sufficient condition* for the property Q and the property Q is a *necessary condition* for the property P.

This double way of expressing a given relation of implication proves to be particularly illuminating. After a little drill in expressing some implications by these two forms, pupils become familiar with the double form.

To the properties P and Q correspond the sets $S(P)$ and $S(Q)$ which contain the elements which satisfy, respectively, P and Q.

If we have $P \rightarrow Q$ then every element which satisfies P, also satisfies Q. Therefore, every element of $S(P)$ belongs to $S(Q)$. We say that $S(P)$ is *contained in*, or is *included in*, $S(Q)$.

We write: $S(P) \subset S(Q)$ [$S(P)$ is contained in $S(Q)$]

To the implication between properties, $P \rightarrow Q$, corresponds inclusion between their extensions (ranges), $S(P) \subset S(Q)$.

If we represent $S(P)$ and $S(Q)$ by the interiors of two circles, as was already done by Euler, we have the following diagram in Figure 1.

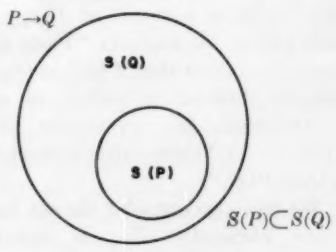


Figure 1

Examples: (A) Saying: "If a quadrilateral is a rectangle, it has equal diagonals" is the same as asserting: "The set of rectangles is contained in the set of quadrilaterals with equal diagonals."

(B) If we multiply the two members of an equation

$$(1) \qquad A(x, \cdots) = B(x, \cdots)$$

by the expression $C(x, \cdots)$, defined for all solutions of the equation but possibly having no solutions, we obtain the equation:

$$(2) \qquad A(x, \cdots)C(x, \cdots) = B(x, \cdots)C(x, \cdots)$$

which is true for all solutions of the first equation.

The first equation, then, implies the second and the set of solutions of (1) is contained in the set of solutions of (2).

V. Equivalence of properties; identity of sets

A. If we have, at the same time, the two reciprocal relations, $P \rightarrow Q$ and $Q \rightarrow P$, each of the properties P and Q is a *necessary and sufficient condition* for the other. They are therefore satisfied in the same cases. We say that they are *logically equivalent* and we write: $P \leftrightarrow Q$.

We have then, between the sets $S(P)$ and $S(Q)$, the two reciprocal inclusions: $S(P) \subset S(Q)$ and $S(Q) \subset S(P)$

Consequently, the sets $S(P)$ and $S(Q)$ have the same elements; they merge into one and the same set: $S(Q) = S(P)$

To the equivalence of two properties corresponds the identity of their extensions.

B. From the logical point of view, the problem of geometric loci of points comes up in three ways.

1. *By definition*, the locus of points satisfying a given condition P is the set of all the points which satisfy this condition.

The geometric locus defined by the condition P of the variable element (point, line, etc. . . .) is then the set $S(P)$. Many loci are introduced by a definition which gives a name to the set $S(P)$ determined by the defining condition P. For example: half plane, circumference, circle, arc of a circle, segment of a circle, etc.

2. Certain *theorems*, such as those on the perpendicular bisector of a line segment and on the angle bisector, assert the identity of the two loci determined by the conditions P and Q and prove their equivalence.

To return to the banal example of the perpendicular bisector of a line segment, we can represent the properties P and Q and the corresponding sets as follows (Figure 2):

$$P: MX \perp AB \qquad Q: AX = BX$$
$$AM = MB$$

$S(P)$: The set of points on the perpendicular erected to AB at its midpoint.
$S(Q)$: The set of points in the plane equidistant from points A and B.

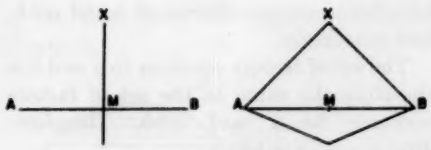


Figure 2

We prove the two reciprocal implications $P \rightarrow Q$ and $Q \rightarrow P$, which is equivalent to proving: $S(P) \subset S(Q)$ and $S(Q) \subset S(P)$. We conclude that $P \leftrightarrow Q$ or $S(P) = S(Q)$.

3. In *finding* a locus, a condition P, defining the locus, is given and we are to find a condition Q equivalent to P.

When the direct proposition $P \rightarrow Q$ or $S(P) \subset S(Q)$ is established, some textbook writers call $S(Q)$ [not $S(P)$] the locus. But this does not fit the definition, or it is premature because we do not know, as yet, whether conversely $S(Q) \subset S(P)$.[2]

C. Two equivalent equations are open sentences which are logically equivalent.

To establish this equivalence, we prove that the corresponding sets are identical by proving that every solution of one equation is a solution of the other, and conversely.

D. In analytic geometry, thanks to a system of co-ordinates, the points of the plane, for example, are put into a one-to-one correspondence with their co-ordinates.

To a geometric locus which is a line, there corresponds an equation which is logically equivalent to the geometric definition of the line.

Therefore, the equivalence of the equations of two loci is the necessary and sufficient condition for the identity of the two loci.

E. In arithmetic, to prove that the integers a and b have the same common divisor as, for instance, a and $a+b$, we prove that every divisor common to a and

[2] Translators' note: Here two paragraphs were included in the original article which are a critique on a method of solving locus problems which is not given in American texts and have therefore been omitted in the translation.

b is also a common divisor of a and $a+b$, and conversely.

The set of factors common to a and b is therefore the same as the set of factors common to a and $a+b$. Therefore: $D(a, b) = D(a, a+b)$.

VI. Conjunction of statements; intersection of sets

A. In conclusion, let us say a few words about two logical operations which permit us to combine propositions and open sentences.

One consists of considering simultaneously two (or more) sentences P and Q. We obtain in this way their *conjunction* "P and Q" which is true only when both propositions are satisfied at the same time.

To the conjunction of propositions P and Q corresponds the *intersection* of the sets $S(P)$ and $S(Q)$, that is, the set of common elements belonging to $S(P)$ and $S(Q)$ at the same time. The intersection of the sets pictured by the circles in Figure 3 is therefore represented by the shaded portion.

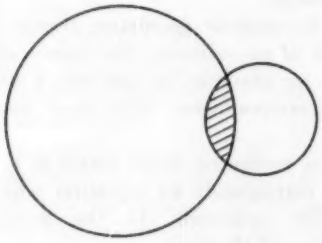


Figure 3

When the sentences P and Q are consistent there exists at least one element which makes their conjunction "P and Q" true. In that case, the intersection of $S(P)$ and $S(Q)$ is not empty. $S(P)$ and $S(Q)$ overlap each other.

When the sentences P and Q are inconsistent or contrary, the intersection of the sets $S(P)$ and $S(Q)$ is empty. We say that such sets are *disjoint*.

For example, two parallel lines or two parallel planes are *disjoint* sets. We can picture disjoint sets by circles external to each other.

B. The method of geometric construction by means of *intersection* of loci is the most powerful one.

Intersections arise also in definitions and in finding new loci from given ones. For example, the acute angle AOB is the *intersection* of two half-planes, as in Figure 4.

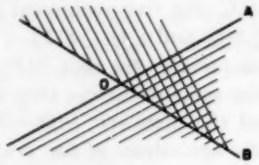


Figure 4

When we are seeking the locus of the midpoints of chords cut off by a circle (O, R) from the secants to the circle from a given point A, it should be noted that the midpoint M of a chord is within the circle (O, R) and that it is on the circumference whose diameter is OA. The student is thus led, by the very analysis of the problem, to take the intersection of these two loci.

C. The system formed by several simultaneous equations (or inequalities) is simply the conjunction of the open sentences that these equations are. The notion of the consistency of several sentences can be applied directly to systems of equations.

D. The graphic solution of systems of equations reduces the determination of the common solutions to finding the intersections of the loci which represent the equations in a system of co-ordinates.

As much can be said also for the graphic solution of inequalities.

E. To give a more general example, let us remark that, since being a square is equivalent to being a rhombus and a rectangle at the same time, the set of squares is the intersection of the set of rhombuses and the set of rectangles. (Here a rhombus is merely an equilateral quadrilateral.)

VII. Disjunction of statements
union of sets

A. A second operation on statements $P, Q, \cdots$, consists of considering the *disjunction* of these statements, that is the statement "P or Q" which is satisfied when at least one of the properties P, Q is satisfied, without excluding the case in which they are both satisfied.[3]

To the disjunction "P or Q" corresponds the *union* of the sets $S(P)$ and $S(Q)$, which is the set of all elements belonging either to $S(P)$ or $S(Q)$. (See Figure 5.)

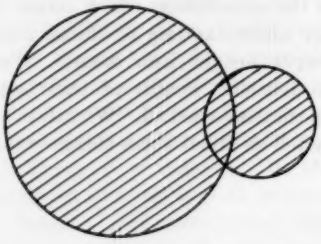


Figure 5

B. (1.) The disjunction of several equations arises in the case of an equation which is reducible.

In fact, when the expressions $A(x \cdots)$ and $B(x \cdots)$ are defined for the solution of the equation $A(x \cdots)\ B(x \cdots)=0$, this equation has for solutions the solutions of the equations $A(x \cdots)=0$ and $B(x \cdots)=0$.

It follows that if the expressions A and B are defined for the solutions of the equation $A^2=B^2$, then this equation reduces to $A=B$ or $A=-B$, and the last two equations can be satisfied simultaneously if $A=0$ and $B=0$.

The set of solutions of $A^2=B^2$ is therefore the *union* of the solutions of $A=B$ and $A=-\mathrm{B}$.

The conjunction of these last equations is satisfied by the solutions of $A=0$ and $B=0$.

[3] Author's note: There is a kind of disjunction, called exclusive disjunction, which excludes the conjunction of the properties P and Q.

We then have the diagram shown in Figure 6.

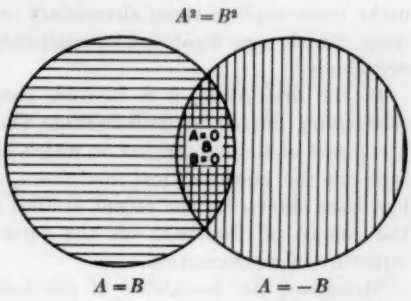


Figure 6

(2.) In the same manner, in analytic geometry, the quadratic equation of two straight lines is the equation of the union of these straight lines.

C. Numerous geometric loci are obtained by the *union* of known loci.

For example, the reflex angle AOB is the union of two half-planes, as shown in Figure 7.

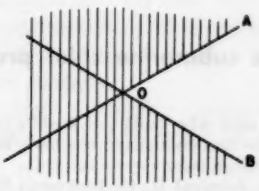


Figure 7

As our pupils frequently are satisfied with giving only one arc when it is a question of finding the locus of points from which a given line segment subtends a given angle, let us teach them to say that the locus is the union of two such arcs.

D. To show the great role that *intersections* and *unions* of sets play in elementary mathematics, it will suffice to recall that the finding of the greatest common factor of several numbers and of their lowest common multiple bring in, respectively, the *intersection* and the *union* of the sets of prime factors of the given numbers.

Conclusion

Why, you will say, is it necessary to make more explicit these elementary notions which are known by practically everybody?

In the first place, it is because these elementary ideas are fundamental. They are a part of the foundation on which are built all the mathematical disciplines, as has been shown by the recent studies of the school of Bourbaki on the formal structure of mathematics.

Moreover, the researches of psychologists in intelligence, in particular those of J. Piaget, show that *inclusion, intersection* and *union of sets* are effectively known long before the properties of whole numbers or the properties of Euclidean geometry which we believe to be elementary.

Finally, all the pedagogic experiments attempted by some of us to make students aware of these ideas have aroused their interest and facilitated their understanding, especially in the study of geometric loci and in equations.

Of course it goes without saying that it is necessary to adapt what we say to the level of the students. To show the full implications of these clarified ideas and to reach all the students, the examples must be taken from everyday life and must be treated in the language of the pupils.

The pupils who evolve in this way some basic structures by their own intelligence have the satisfaction which comes from a better understanding of things and from a deeper insight into oneself. We have often felt the weight of their reproach when they have asked, "Why didn't somebody tell us about these things before?"

Atomic submarine chief proposes new school system

The man who built the world's first atomic submarine has challenged the basic structure of the American school system.

Rear Admiral H. G. Rickover, USN, Chief of the Naval Reactors Branch of the Atomic Energy Commission, charged that, "To put it bluntly, our schools do not perform their primary purpose which is to train the nation's brain power to the highest potential. . . . We shall not do justice to our talented youth until we seek them out at an early age—no later than ten or eleven—and educate them separately from the rest of the children." This should be done, he said, "preferably in separate schools or, if this is not possible, in separate classes."

Pointing out that "there is no general agreement on the definition of the term "talented," Admiral Rickover used "talented" to mean the top 15–20 per cent of our children and "brilliant" to mean the top 1½–2 per cent.

Speaking before more than three hundred educators, industrialists, and government officials at the Seventh Thomas Alva Edison Foundation Institute on science education, Admiral Rickover stated that, "For the last fifty years we have, in the name of educational democracy, tried to make one common school serve all children instead of finding the appropriate school for the two main groups of children—the majority who plan on non-academic careers, and the minority who plan for college and university. We must reverse this unfortunate trend in American education.

"I suggest that industry, together with our educational foundations undertake the setting up of model academic secondary schools in perhaps twenty-five different centers in the United States. These schools would be on a par with our best academic secondary schools. They would be free; the ability to pass an entrance examination of a kind which would weed out those not mentally capable of absorbing an academic secondary education would be the only requirement. The schools would be staffed by teachers truly capable of teaching talented children; teachers whose qualifications place less emphasis on training in teaching *methods* and more on graduate study in their fields.

"Teachers' salaries in these schools would be in accord with the high scholastic qualifications required and, therefore, equal to those paid for comparable positions in industry.

Continued on page 124

A study on the use of science counselors

JOHN R. MAYOR, *American Association for the Advancement of Science, Washington, D. C.*

The efforts of various groups to organize for an attack on the various problems of instruction are indicative of intense interest in the status of secondary mathematics and secondary science.

THE UNIVERSITIES of Nebraska, Oregon, and Texas, and Pennsylvania State University have been chosen to serve as centers for the Study on the Use of Science Counselors during the school year 1956–57. This study is one of the major activities of the Science Teaching Improvement Program of the American Association for the Advancement of Science. The study was developed as a result of an awareness of the present and anticipated future shortage of science and mathematics teachers. It is recognized that we face in the next ten years a serious shortage of teachers at all levels of instruction and that the shortage will be especially critical in science and mathematics because of the job opportunities in industry for persons trained in these subjects. It is further recognized that the nature of science and mathematics is such that teachers in secondary schools, no matter what their previous training or ability, have real difficulty in keeping abreast of modern developments in content and curriculum.

The study is also based on the recognition of the valuable service that can be provided to teachers by competent supervisors and counselors. In American education the use of supervisors is far too limited, particularly in science and mathematics. It is believed that a science counselor, with breadth and depth of training in the sciences and mathematics, a background of outstanding accomplishments in teaching, an ability to work with people, and a practical knowledge of the learning process, can stimulate and guide the work of a small group of less well-prepared and less-experienced teachers.

The Study on the Use of Science Counselors is planned to test a method of:

1. Improving the science and mathematics instruction in secondary schools through the use of teacher counselors.
2. Providing in-service aid to teachers of science and mathematics, with stress on most recent developments, improved laboratory and demonstration techniques, and good teaching methods.
3. Bringing university staff members in science, mathematics, and education into closer working relationships on a problem of common concern.
4. Bringing university science and mathematics staffs into closer touch with secondary schools.
5. Providing direct service by universities to secondary schools.
6. Providing a means of increasing the co-operation between staff members of universities and the state departments of education.

Each of the four centers for the study consists of a state university and the territory surrounding it, so bounded as to contain as many secondary schools cooperating in the project as will be required to include approximately one hundred full-time science and mathematics teachers. The centers may not include a high school or school system which employs a science supervisor, counselor, or consultant on either a part-time or full-time

basis. It has been planned that one center should be in the West, one in the South, one in the Midwest, and one in the East. In each of the participating universities the study will have the support of the science, mathematics, and education departments. The co-operating institutions have designated committees of staff members, with the center co-ordinators as chairmen, to act in an advisory capacity to the Center Study and to the counselors. The universities are responsible for establishing satisfactory working relationships with the co-operating secondary schools and for obtaining the approval and co-operation of the state department of education in the study.

Elizabeth Roudebush, Director of Mathematics, Seattle Public Schools, and John A. Brown, State University Teachers College, Oneonta, New York, served as mathematics consultants for the University of Colorado Conference, held in June 1956, in preparation for the study. John R. Mayor is Director of the study. A pamphlet describing the study may be obtained on request from the American Association for the Advancement of Science, 1515 Massachusetts Avenue, N. W., Washington 5, D. C.

Atomic submarine chief

Continued from page 122

"These model schools should aim at a ratio of at least one teacher for every twenty pupils (and) would start with the fifth grade so as to have the pupils ready for college at sixteen. We must find a way for our children to attend school for more than the present 180 days. Lengthening the time in school might most easily be done through the medium of voluntary summer courses.

"I estimate that the cost of operating each model school for a period of five years will be about ten million dollars. At the end of that period the community ought to have an option to take over the school, provided it agrees to continue the high scholastic standards set under private management.

"We must not forget that well-to-do parents always have it in their power to assure their children a good education by sending them to private preparatory schools. But the talented *poor* child must depend solely on the public school. Education in a democracy must not only be democratic, it must also be education."

Admiral Rickover said that these educational reforms are especially needed because of the critical shortage of scientists and engineers in the United States. "Trained manpower has become a weapon in this cold war since it determines how much technical aid each side can offer. Russia will soon have a *surplus* of trained manpower which she can export. We have a *shortage*. Can we allow Russia to outdo us in aiding the neutrals? Can we let her become Big Brother to all the backward countries of the world?

"No matter what methods are chosen in the duel between the communist and the free world, education will, in the final analysis, determine the outcome, particularly the education of talented youth. . .

"Our social mores are hostile to the concept that children of superior mentality ought to receive special consideration at tax-supported schools. The very thought of recognizing differences in intellectual ability is repugnant to our equalitarian philosophy. . . . We are committed to the basic assumption that there is no person who can claim to be an *indispensable man*. We proceed from this entirely correct assumption to the incorrect conclusion that neither does a democracy have *indispensable men*. This is obviously erroneous . . . no society can function without its indispensable men."

"Today, technological progress is limited only by availability of trained professionals, and this, in turn, depends on but two factors: incidence of superior brain power—over which we have little, if any, control; and development of available talent—here we can do much."

The Seventh Edison Foundation Institute dealt with "Strengthening Science Education for Youth and Industry." Prominent educators, businessmen, and government officials took part in a series of panel discussions concerning various aspects of this problem.

Held at Glenmont, the home of Thomas Alva Edison in West Orange, New Jersey, and the Hotel Suburban in East Orange, the two-day conference was also addressed by Charles F. Kettering, President of the Edison Foundation; Sherrod E. Skinner, Vice President, General Motors Corporation; Walker L. Cisler, President of The Detroit Edison Company; and Dr. Herbert Scoville, Jr., Assistant Director, U. S. Central Intelligence Agency.

New exercises in plane geometry

MATHEMATICS STAFF OF THE COLLEGE,
University of Chicago, Chicago, Illinois.

Previous discussions of the cutting of squares suggest geometrical exercises of a new and intriguing type. Certain exercises of this type are solved here. The article supplements material in the same vein in The Mathematics Student Journal *for February* 1957.

OUR THREE PRECEDING PAPERS[1] on the theme of cutting squares saw the solution of certain standard problems: to transform three congruent squares into a single square (Problem I), and conversely to transform a single square into three congruent squares (Problem II); to transform any two squares into a single square (Problem III); to transform a square into an equilateral triangle (Problem IV), and conversely to transform an equilateral triangle into a square (Problem V); to transform a nonrectangular parallelogram into a square (Problem VI); to transform a square into two or more congruent equilateral triangles (Problem VII); and finally, to transform one or more rectangles into a square (Problem VIII).

Between them, these eight problems suggest an approach to any such transformation problem and comprise a variety of methods useful in connection therewith. With the help of these eight problems and their solutions we have gone some way toward reducing the indeterminateness of any such problem.[2]

Now, it is an easy matter to invent more problems of this sort. What is less easy is to invent problems which are not too hard, on the reasonable ground that it is better to solve easy problems than to contemplate difficult ones. Still less easy is it to invent problems whose solutions have implications beyond the particular problems and which (when solved in a certain order) succeed in suggesting alternative solutions to one or another of the eight problems we have already presented.

The seven exercises presented and solved below are chosen with these conditions in mind. All of them relate to our general theme of cutting squares. None of them requires extensive background. (Indeed, while the solution of Exercise 1 relies on our treatment of Problem II, *the remaining six exercises are independent of our previous papers and can be read without knowledge of them.*) All are comparatively easy. And Exercises 3, 4, and 5 suggest alternative solutions of our Problem VIII, one such alternative appearing in Exercise 6.

The material of this paper is based in part on portions of the book *The Wonders of the Square* by B. Kordiemski and N. Rusalev.

EXERCISE 1 AND ITS SOLUTION

We begin with an exercise which resembles Problem II, the problem of transforming a square into three congruent squares.

[1] "A Problem on the Cutting of Squares," "More on the Cutting of Squares," and "Still More on the Cutting of Squares," THE MATHEMATICS TEACHER XLIX (May, October, December 1956), 332–43, 442–54, 585–96.

[2] In this connection, recall what we said under the heading "Some Pedagogical Remarks" in our first paper.

Exercise 1: *Given a square, to transform it into five congruent squares.*

Solution: It would seem evident that this exercise can be solved in essentially the same way as was Problem II in our first paper, where we gave three solutions. Let us follow the method of the *second* of these solutions.

To begin, suppose our given square $ABCD$ in Figure 1 has a side of length 1. First we want to locate on DC extended a point E such that $(AE)=\sqrt{5}$. [Recall our convention that '(XY)' designates the length of segment XY.] For this, it is sufficient to locate E such that $(DE)=2(DC)=2$, since then

$$(AE)=\sqrt{(AD)^2+(DE)^2}=\sqrt{1+4}=\sqrt{5}$$

From this point, the steps in the construction and proof are analogous to those of the second solution of Problem II. Following the steps of the construction (shown in Figure 1) and making some easy computations, we conclude that the transformation of a square into five congruent squares requires four (straight) cuts: along segments AG, CL, DK, and BN respectively. Here AG is parallel to CL, DK is parallel to BN, and K, G, N, and L are respectively the midpoints of the sides of our square. In Figure 1, the triangle labelled '1' is common to $ABCD$ and the rectangle whose side BF is five times longer than its adjacent side FH. The other parts of this rectangle are labelled with the same numerals as their congruent counterparts in $ABCD$.

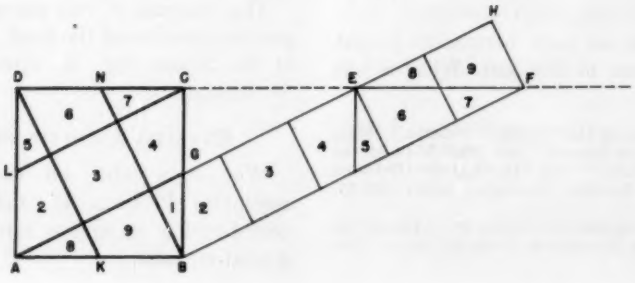
Remarks: (a) This solution sees the division of the given square $ABCD$ into *nine* parts: four congruent trapezoids (Parts 2, 4, 6, 9), four congruent right triangles (Parts 1, 5, 7, 8), and one square (Part 3) whose area is one-fifth that of $ABCD$. If this small square (Part 3) is cut into a trapezoid and a triangle congruent respectively to, e.g., Part 2 and Part 1, we obtain ten parts all told—five congruent trapezoids and five congruent triangles. These ten parts can be made the basis of a puzzle: *Given the ten parts, to form five congruent squares; and to form one square.* (b) Recall the *third* solution of Problem II in our first paper. The method of that solution can also be used here. The resulting solution sees square $ABCD$ cut into *ten* parts, not nine or fewer. (You might verify this.) What is surprising here is that the third method *increases*, rather than decreases, the number of parts, exactly opposite to Problem II. (c) Is it possible to adapt the first method of solution of Problem II to our present Exercise 1?

Exercise 2 and its solution

Exercise 2: *Given a square, to transform it into eight congruent squares.*

Solution: This exercise is not particularly

Figure 1



intricate. Suppose the given square is *ABCD*. Cutting this square along its diagonals *AC* and *BD*, and labelling the center '*O*', we obtain four congruent right triangles *AOB*, *BOC*, *COD*, and *DOA* as shown in Figure 2. From these four triangles two congruent squares can be formed. Each of these squares can readily be divided into four congruent squares, whence we obtain eight congruent squares as desired. The whole solution is made very explicit in Figure 2.

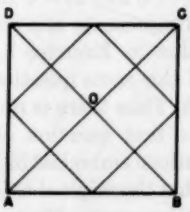


Figure 2

Remarks: The first step was to transform the given square into two congruent squares. The simple transformation devised for the purpose (viz., cut the original square along each diagonal) *seems* to be much simpler than the one devised previously for the same purpose. (See our remarks in connection with the converse of Problem III, in the second paper of this series.) But is it, actually?

Exercise 3 and its solution

It is claimed that Galileo wrote a mathematical book called *Elim*[3] in which reference is made to a problem posed to his contemporary, the physician and mathematician Del Medice, by the Arabian savant Ali of Cairo: "We have a rectangular board measuring 5 by 2; compose from this rectangle a square by cutting the board into exactly four parts." It may be wondered why this sixteenth century Arab asked for four parts, rather than three; he may have thought a 5 by 2 rectangle could not be divided into three parts that reassemble into a square, but in this case he was mistaken.

[3] The claim is made by Kordiemski and Rusalev on p. 50 of their book *The Wonders of the Square*. European and American students of Galileo appear not to know of *Elim;* perhaps it is a recent discovery by the Russians.

Exercise 3: *Given a rectangle measuring 5 by 2, to cut it into four parts that reassemble into a square; to cut it into three parts that reassemble into a square.*

Solution: Strictly speaking, Exercise 3 is a particular case of the general problem of transforming a rectangle into a square. Thus one approach to a solution could be the method we devised for the general Problem VIII. However, it is often quite suggestive to proceed, not from the general to the particular, but from the particular to the general—from the special case to the general one, from a device effective for a particular problem to a method for all analogous problems. Let us therefore ignore our general solution (Problem VIII) and confront Exercise 3 directly.

Picking a convenient unit, let us construct a rectangle *ABCD* such that $(AB) = (CD) = 2$ and $(AD) = (BC) = 5$. What we seek is a transformation of *ABCD* into a square of side $\sqrt{5 \cdot 2}$, i.e., a side $\sqrt{10}$. It is easy to construct a segment of length $\sqrt{10}$; e.g., the hypotenuse of a right triangle with legs 1 and 3 respectively is a segment of length $\sqrt{10}$. On side *AD* of our rectangle mark the point *E* such that $(AE) = \sqrt{10}$, and on side *BC* mark the point *K* such that $(DK) = \sqrt{10}$. Draw segment *DK*. Next, on *BK* mark the point *F* such that $(KF) = (DE)$, and draw segment *EF*. Finally, draw segment *KM* perpendicular to *EF*. This construction divides our original rectangle *ABCD* into four parts (Parts 1, 2, 3, 4 of Figure 3 on page 128); we shall show that these four parts reassemble into a square.

Because its sides *DE* and *KF* are congruent and parallel, the quadrilateral *EFKD* in Figure 3 is a parallelogram. Therefore $(EF) = (DK) = \sqrt{10}$ and *EF* is parallel to *DK*; and further, $\angle AEF = \angle ADK$. In view of the fact that $\angle ADK + \angle KDC = 90°$, we therefore know

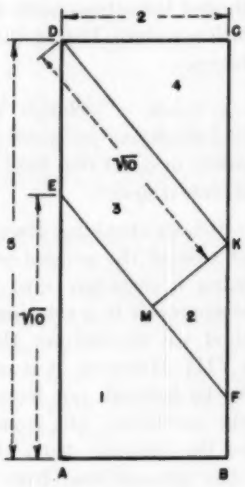


Figure 3

that $\angle AEF + \angle KDC = 90°$. Hence, if we place trapezoid $AEFB$ (Part 1) against triangle DCK (Part 4) in such a way that their respective sides EF and DK are coextensive, we obtain a rectangle with sides $\sqrt{10}$ and 2. This rectangle is shown as rectangle $LNPQ$ in Figure 4; its sides LN and LQ are such that $(LN) = (AE) = \sqrt{10}$, and $(LQ) = 2$.

From similar considerations it follows that, by laying triangle FMK (Part 2) against trapezoid $EMKD$ (Part 3) so that sides FK and ED are coextensive, we can obtain another rectangle with side $\sqrt{10}$. This rectangle appears in Figure 4 as $QRTP$, and $(RT) = (KD) = \sqrt{10}$. How long is QR, the second side of $QRTP$? To compute its length, let us refer to Figure 3 and note that $(QR) = (MK)$. Now the right triangles FMK and KCD in Figure 3 are similar, because $\angle MFK = \angle CKD$; therefore

$$\frac{(MK)}{(KF)} = \frac{(DC)}{(DK)}.$$

Since $(KF) = (DE) = 5 - \sqrt{10}$, we see that

$$(MK) = \frac{(KF)\cdot(DC)}{(DK)} = \frac{(5-\sqrt{10})\cdot 2}{\sqrt{10}}$$

$$= \frac{(5-\sqrt{10})\sqrt{10}}{5} = \sqrt{10} - 2.$$

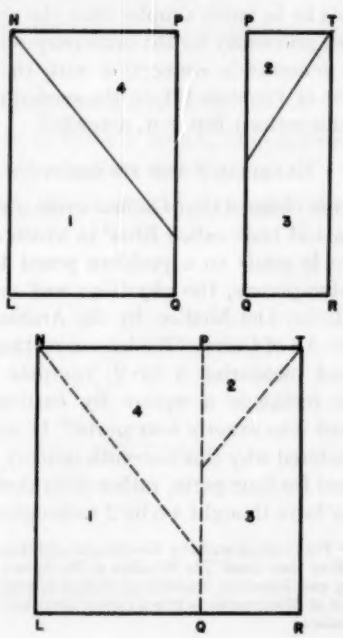
Consequently $(QR) = (MK) = \sqrt{10} - 2$, and our rectangle $QRTP$ has sides $\sqrt{10}$ and $\sqrt{10} - 2$ respectively.

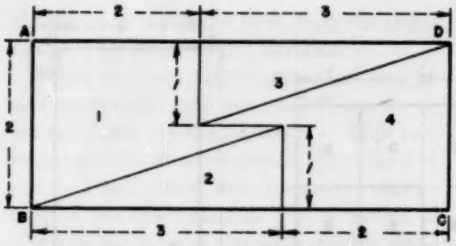
Thus, placing rectangles $LNPQ$ and $QPTR$ against each other with the side QP in common, we obtain the rectangle $LNTR$ as shown in Figure 4; and $LNTR$ is the square desired, for $(LR) = (LQ) + (QR) = 2 + (\sqrt{10} - 2) = \sqrt{10} = (RT)$.

Figure 3 represents one solution of the first question in Exercise 3. A different solution of this same question is disclosed in Figure 5. Thus there is no unique solution to the first question of Exercise 3.

The solutions embodied in Figures 3 and 5 require that the original rectangle $ABCD$ be cut into four parts. We turn to the second question in Exercise 3, that of dividing rectangle $ABCD$ into three parts that reassemble into a square.

Figure 4





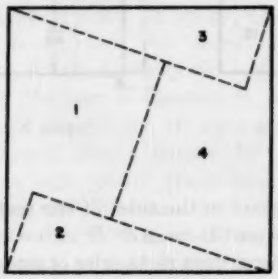


Figure 5

The construction, shown in Figure 6, is quite simple. On the long sides AD and BC of our given rectangle, mark respectively the points E and F such that $(AE)=(CF)=\sqrt{10}$. Draw segment DF, and draw segment EK parallel to DC. The three parts into which this construction divides $ABCD$ can be reassembled into a square, as we shall now show. Evidently $(DE)=(FB)$ and $\angle EDK=\angle DFC$. Therefore, putting against the pentagon $AEKFB$ (Part 1) the triangle EDK (Part 2) so that the congruent sides DE and FB are coextensive, we obtain a trapezoid which fails of being a complete square by precisely the amount of the right triangle DCF (Part 3).

Remarks: Can you find a solution of the three-part question different from that expressed by Figure 6? What ratio must the length of the long side of a rectangle bear to that of its short side if the method expressed in Figure 6 is to succeed in transforming the rectangle into a square? How can this method be generalized to apply to any rectangle (see Exercise 6)?

Exercise 4 and its solution

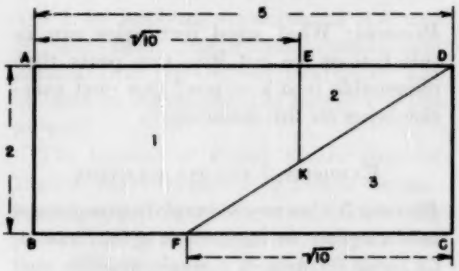
In Exercise 3 we were required to compose a square out of four (or even three) parts of a certain sized rectangle. Exercise 4 has a similar character.

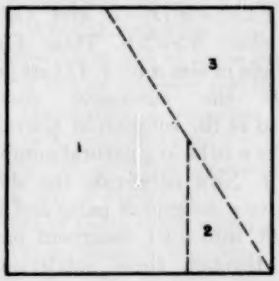
Exercise 4: *Given a rectangle measuring 9 by 16, to cut it into two parts that reassemble into a square.*

Solution: Choosing a convenient unit, let us construct a rectangle $ABCD$ such that $(AB)=(CD)=9$ and $(AC)=(BD)=16$. The area of $ABCD$ is therefore *144* square units. The area of the square sought for is then also 144, and its side has a length of 12. Which is to say, the desired square must have a side *three* units longer than the smaller side of $ABCD$, and *four* units shorter than the longer side of $ABCD$.

Divide side AB into three congruent parts, and side BC into four congruent parts, and through the points of subdivision draw lines parallel to the sides of

Figure 6





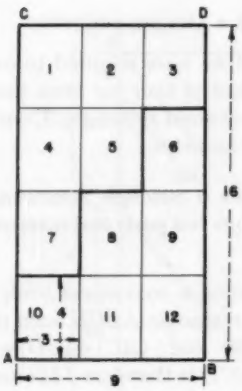


Figure 7

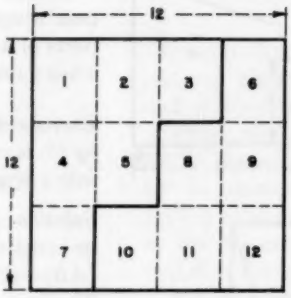


Figure 8

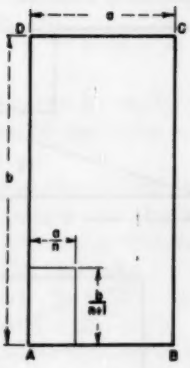


Figure 9

the rectangle. As Figure 7 shows, this construction partitions rectangle $ABCD$ into four rows each comprising three congruent rectangles measuring 3 by 4.

To obtain a square from Figure 7, it is sufficient to rearrange these small rectangles into three rows of four rectangles each. A moment's reflection convinces us that the staircase cut shown in Figure 7 does the trick. The resulting square is shown in Figure 8.

Remarks: What sized rectangles can in this fashion be cut into two parts that reassemble into a square? Our next exercise bears on this question.

Exercise 5 and its solution

Exercise 5: Can any rectangle be transformed into a square in the fashion of Exercise 4, i.e., with the help of a single staircase cut?

Solution: Let $ABCD$ be a rectangle such that $(AB)=(CD)=a$ and $(BC)=(AD)=b$, where $b>a>0$. Thus $ABCD$ is a rectangle of size a by b. Let us perform on $ABCD$ the successive constructions utilized in the solution of Exercise 4.

Take n to be any natural number greater than 1. Now subdivide the shorter side AB into n congruent parts and the longer side BC into $n+1$ congruent parts. Next, draw through these subdivision points lines parallel to the sides of the rectangle; this step partitions $ABCD$ into $n\cdot(n+1)$ smaller congruent rectangles of size a/n by $b/n+1$, as indicated in Figure 9. Finally, divide our rectangle $ABCD$ by the staircase cut and transpose the resulting two parts in the fashion of Exercise 4.

The figure we obtain after this transposition is obviously a new rectangle of size

$$a+\frac{a}{n} \quad \text{by} \quad b-\frac{b}{n+1}\,.$$

This new rectangle will be a square provided

$$a+\frac{a}{n}=b-\frac{b}{n+1}\,.$$

Multiplying the equation through by $n\cdot(n+1)$ and simplifying, we obtain

$$a\cdot(n+1)^2=bn^2,$$

and so see that

$$\frac{b}{a}=\frac{(n+1)^2}{n^2}\,. \tag{1}$$

Equation (1) tells us that our original rectangle $ABCD$ can be transformed into a square by a single staircase cut *just in case* there is a natural number n, $n>1$, such that the length of the longer side of our rectangle bears to the length of its

shorter side the same ratio that the number $(n+1)^2$ bears to the number n^2.

What are some rectangles that can be so transformed into squares? When $n=2$, we see from (1) that $b/a=9/4$; thus any rectangle whose sides are in the ratio 9:4 can be transformed into a square with the help of one staircase cut. The rectangle of size 4 by 9 can be so transformed; the same is true of the rectangle sized 8 by 18, of the rectangle sized 12 by 27, etc. Again, when $n=3$, we see from (1) that $b/a=16/9$; consequently rectangles of size 9 by 16 (the case of Exercise 4), of size 18 by 32, of size 27 by 48, etc., all can be transformed into a square by a single staircase cut. Other transformable rectangles of still different dimensions can be obtained with the help of (1) by setting n in turn equal to 4, to 5, etc.

Is there a rectangle $ABCD$ whose dimensions a and b are such that it *cannot* be transformed into a square with the help of a single staircase cut? This question amounts to asking: Can we pick dimensions a and b such that *no* natural number n greater than 1 satisfies equation (1)? Yes. Consider a rectangle $ABCD$ of size 1 by 2, i.e., a rectangle such that $b/a=2$. By (1), this rectangle transforms into a square under one staircase cut just in case there is a natural number n such that

$$\frac{(n+1)^2}{n^2}=2;$$

which is to say, just in case there is a natural number n such that

$$n^2-2n-1=0.$$

But the solutions of this quadratic equation are *not* natural numbers. Hence *no* rectangle whose sides are in the ratio 2:1 can be transformed into a square by one staircase cut.

Remarks: The question which constitutes Exercise 5 has now been answered in the negative. Also, equation (1) furnishes a complete answer to the question raised after our solution of Exercise 4.

Exercise 6 and its solution

Our solutions of Exercises 3, 4 , and 5 show that rectangles of certain particular dimensions can be transformed into squares with the help of three cuts, or two, or even one. Any one of these methods of solution can, *with small improvements*, be used to transform an *arbitrary* rectangle into a square. Exercise 6 which follows requires us to adapt one of these methods to the general case.

Exercise 6: *Adapt the method expressed in Figure 6 (for a rectangle of size 2 by 5) to a rectangle of arbitrary dimensions.*

Solution: Let us begin with an arbitrary rectangle of size a by b, viz., the rectangle $ABCD$ such that $(AB)=(CD)=a$ and $(BC)=(AD)=b$ and $0<a<b$. The area of this rectangle $ABCD$ is $a\cdot b$, hence a square with the same area must have a side of length $\sqrt{a\cdot b}$.

In following out the method expressed in Figure 6, we need a segment of length $\sqrt{a\cdot b}$ But the construction of such a segment presents no problem; $\sqrt{a\cdot b}$ is the mean proportional between a and b, hence we readily obtain a segment of length $\sqrt{a\cdot b}$ by applying to segments AB and BC the mean proportional construction familiar from our October 1956 paper. Let us suppose this segment of length $\sqrt{a\cdot b}$ is at hand.

The method of Figure 6 now requires that we mark on each long side of our rectangle a certain point—on side AD, the point E such that $(AE)=\sqrt{a\cdot b}$; and on side BC, the point F such that $(CF)=\sqrt{a\cdot b}$. The next step is to draw segment DF, and to construct segment EK parallel to side AB. When this step has been taken, we find ourselves in one of three possible situations: K lies on segment DF between D and F (as shown in diagram (i) of Figure 10); or K coincides with F (diagram (ii) of Figure 10); or K lies between F and B (diagram (iii) of Figure 10).

Which one of these three situations we find ourselves in depends, of course, on the dimensions of our original rectangle

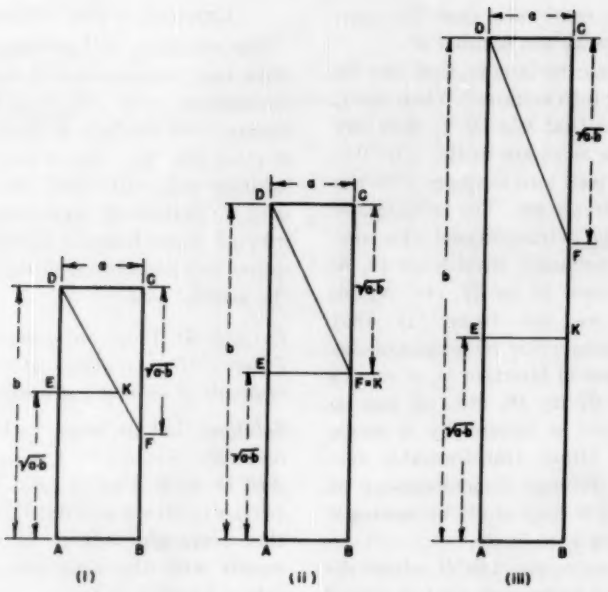


Figure 10

ABCD. For point *K* lies *on* segment *DF* just in case the length of segment *AE* is *not* less than half the length of side *AD*, *i.e.*, just in case

$$\frac{(AD)}{2} \leqq (AE).$$

Since $(AD)=b$ and $(AE)=\sqrt{a \cdot b}$, this inequality amounts to

$$\frac{b}{2} \leqq \sqrt{a \cdot b}.$$

Squaring, we have

$$\frac{b^2}{4} \leqq a \cdot b,$$

whence

(2) $$b \leqq 4a.$$

We see from inequality (2) that: point *K* lies on *DF* between *D* and *F* *provided* $b<4a$; point *K* coincides with point *F* *provided* $b=4a$; and point *K* lies between *F* and *B* *provided* $b>4a$.

We must show that, no matter which of the three possibilities eventuates, the method now in use is effective in transforming rectangle *ABCD* into a square.

Consider the first case: *K* falls on *DF* between *D* and *F*, i.e., $b<4a$. As we see from diagram (i) of Figure 10, this situation is completely analogous to that of Figure 6. Thus the transformation into a square of any rectangle *ABCD*, with dimensions *a* and *b* such that $b<4a$, is the familiar one shown in Figure 11. (Note that since $5<4 \cdot 2$, Figure 6 is simply a particular instance of Figure 11.)

Next, consider the second case: *K* coincides with *F*, i.e., $b=4a$. This situation is pictured in diagram (ii) of Figure 10. Here, since $b=4a$, we see that $\sqrt{a \cdot b} = \sqrt{4a^2}=2a$. Hence

$$(AE)=(CF)=\sqrt{a \cdot b}=2a=2(AB).$$

Further $(EK)=(AB)$. The two rectangles *ABFE* and *EFCD* therefore have the same dimensions, *a* by 2*a*. Putting these two rectangles together so that sides *BF*

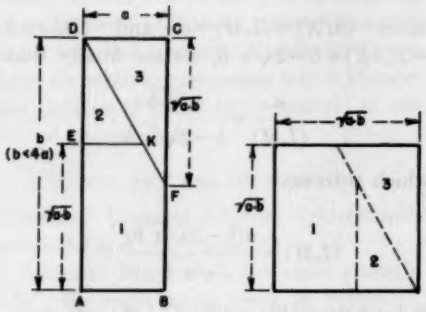
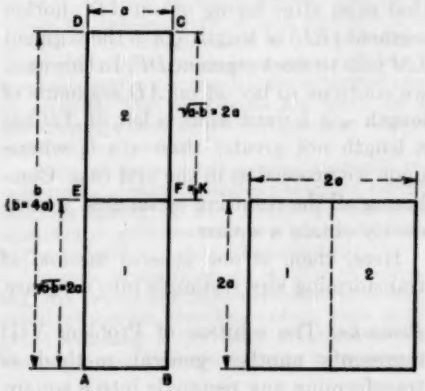


Figure 11

and ED are coextensive, there results a square (of size $2a$ by $2a$) as desired. In this case the cutting of rectangle $ABCD$ along segment DE is evidently unnecessary. The transformation for the second case is shown in Figure 12.

Finally, consider the third case: K falls between B and F, i.e., $b>4a$. This situation is shown in diagram (iii) of Figure 10. Note that here EK does not intersect DF, hence $(EK)=a$ and $ABKE$ is a rectangle of size a by $\sqrt{a\cdot b}$. Now let us proceed as follows: Cut off from rectangle $ABCD$ the smaller rectangular part $ABKE$; considering just the remaining rectangle $EKCD$, locate on ED the point L such that $(EL)=(AE)=\sqrt{a\cdot b}$ and locate on KC the point F such that $(CF)=\sqrt{a\cdot b}$; lastly,

Figure 12



draw segment DF and draw segment LM parallel to EK (hence parallel to side AB). As we know from previous work, point M takes one of three positions depending on the dimensions of $EKCD$: M falls on DF between D and F; M coincides with F; M falls between F and K. Each of these possible situations will now be discussed.

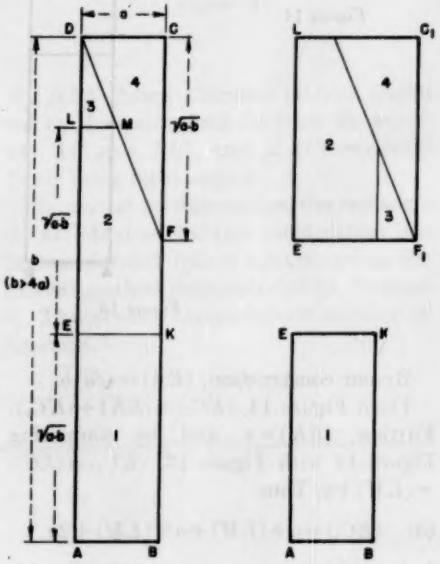


Figure 13

Suppose, *first*, that M falls on DF between D and F. In this case, as Figure 13 makes plain, the parts *2*, *3*, *4* of $EKCD$ reassemble into a rectangle EF_1C_1L. Since $(LE)=\sqrt{a\cdot b}$ and $(LC_1)=(LM)+(DC)=(LM)+a$, this new rectangle has dimensions $(LM)+a$ by $\sqrt{a\cdot b}$. Lay rectangle EF_1C_1L beside rectangle $ABKE$ (Part 1 of $ABCD$) in such a way that side BK is coextensive with side EL. The result is the new rectangle AF_1C_1E shown in Figure 14. This new rectangle AF_1C_1E is in fact a square, as desired.

By way of verification, let us compute the lengths (EA) and (EC_1) of the sides of AF_1C_1E.

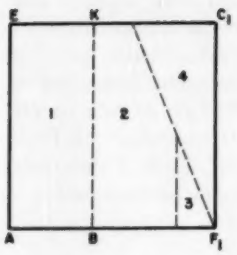


Figure 14

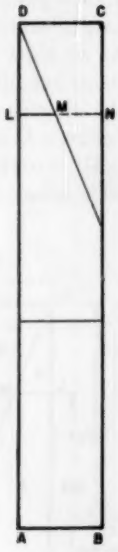


Figure 15

By our construction, $(EA)=\sqrt{a\cdot b}$.

From Figure 14, $(EC_1)=(EK)+(KC_1)$. Further, $(EK)=a$; and, by comparing Figure 14 with Figure 13, $(KC_1)=(LC_1)=(LM)+a$. Thus

(3) $(EC_1)=a+(LM)+a=(LM)+2a$.

Let us therefore compute (LM). To this end, add to rectangle $ABCD$ of Figure 13 and obtain Figure 15 as follows: extend LM to the point N on side BC. Now triangles MNF and MLD are evidently similar, and so

$$\frac{(MN)}{(LM)}=\frac{(FN)}{(DL)}.$$

Observing that $(FN)=(FC)-(NC)=(FC)-(DL)=\sqrt{a\cdot b}-(DL)$, we have

$$\frac{(MN)}{(LM)}=\frac{\sqrt{a\cdot b}-(DL)}{(DL)}.$$

On the basis of a familiar property of ratios, this last equality yields

$$\frac{(MN)+(LM)}{(LM)}=\frac{\sqrt{a\cdot b}-(DL)+(DL)}{(DL)}.$$

Since $(MN)+(LM)=a$ and $(DL)=b-2(AE)=b-2\sqrt{a\cdot b}$, we see finally that

$$\frac{a}{(LM)}=\frac{\sqrt{a\cdot b}}{b-2\sqrt{a\cdot b}},$$

which is to say

$$(LM)=\frac{a(b-2\sqrt{a\cdot b})}{\sqrt{a\cdot b}}.$$

Substituting this value of (LM) into equation (1) we obtain

$$(EC_1)=\frac{a(b-2\sqrt{a\cdot b})}{\sqrt{a\cdot b}}+2a$$

$$=\frac{a(b-2\sqrt{a\cdot b}+2\sqrt{a\cdot b})}{\sqrt{a\cdot b}}$$

$$=\frac{a\cdot b}{\sqrt{a\cdot b}}=\sqrt{a\cdot b},$$

the value of (EC_1) desired.

Hence $(EC_1)=(EA)$, and rectangle AF_1C_1E is indeed a square.

The discussion above makes apparent what happens in the *second* case, when M coincides with F. We leave the details of this case to the reader, remarking only that the square finally produced has the dimensions $3a$ by $3a$.

Should the *third* case arise, viz., the case in which M falls between FK, it means the larger side of rectangle $ABCD$ is so much greater than its smaller side that even after laying off on AD another segment (EL) of length $\sqrt{a\cdot b}$ the segment LM fails to meet segment DF. In this case, we continue to lay off on AD segments of length $\sqrt{a\cdot b}$ until what is left of AD has a length not greater than $\sqrt{a\cdot b}$, whereupon we proceed as in the first case. Combining all the resulting rectangles, we evidently obtain a square.

Here, then, is one general method of transforming any rectangle into a square.

Remarks: The solution of Problem VIII represents another general method of transforming any rectangle into a square.

There are still others; can you develop one of your own? (All these general methods have for their purpose simply to transform an arbitrary rectangle into a square; the number of parts the rectangle is cut into is disregarded.)

Exercise 7 and its solution

Exercise 7: Given an arbitrary right triangle, to transform it into a square.

Solution: Begin with the right triangle ABC, the right angle being at vertex C. Cut this triangle along the segment ED of midpoints that is parallel to side CA. Transposing triangle BED to the position AFD, as shown in Figure 16, we obtain a rectangle.

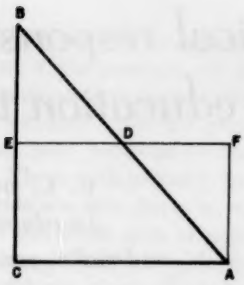


Figure 16

For the proof, let us construct on segments AC and EC the rectangle $AFEC$ and show that triangles AFD and BED are congruent. But this last is an immediate consequence of the following facts: segments AF and EC and also BE are all congruent (by construction); $\angle FAD = \angle DBE$ (being alternate interior angles cut by the transversal AB from the parallels AF and BE); and $\angle AFD = \angle DEB$ (both being right angles).

No matter its dimensions, the rectangle $AFEC$ obtained by this construction can be transformed into a square (using the general method demonstrated in Exercise 6). This remark completes our solution of Exercise 7.

Have you read?

McAulay, J. D., "The Screening of Teachers," *Peabody Journal of Education*, September 1956, pp. 93–97.

As members of a profession we are all interested in securing for our profession only those whom we can be proud of and who will contribute to the advancement of the profession. This article reports a study of attitudes toward screening and types of screening activities now in progress. The conclusions are based on a questionnaire sent to student teachers and to public school teachers.

Some of the conclusions are: academic requirements should be higher; teachers college faculties should do the screening; screening should be done before the senior year is reached, and social stability should be considered.

In spite of the teachers' reaction to screening, the twenty institutions studied have very little continuous screening. Most have some grade standard, a few have committees, and some rely upon individual faculty members. As members of the profession, we should each read the article and then consider what we should be doing about it.

Nemetz, Anthony, "On the Teacher," *Educational Research Bulletin*, Ohio State University, September 12, 1956, pp. 154–63.

Here is an article that will not leave you complacent. How do you define teaching and how do you allocate the effect of environment and the teacher? Do you think teaching is a democratic or autocratic process? Does every man have a right to his opinion? Why is the realm of truth a monarchy?

The author answers these questions and then sets up the teaching conditions and likely results. For example, we would have attentive, teachable, good-willed students. There would be investigations, cross examinations, personal discovery, and creative thinking.

I can't agree in full, but the article still has its points.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Critical responsibilities in education today

B. L. DODDS,[1] *University of Illinois, Urbana, Illinois.*

An educator looks at mathematics in the secondary schools and makes some observations concerning present day conditions.

I HAVE HAD a particular interest in the curriculum problems in the fields of mathematics and science because of some high school teaching experience in these fields, but that was more years ago than I care to remember. In this presentation there will be no attempt to consider the precise problems of curriculum organization and instruction in mathematics, with which you as active workers in the field are more competent to deal than I. Rather, it seems more appropriate to present some observations about the current scene in secondary education and hazard some further observations about implications for the development and improvement of the mathematics program in schools and particularly in secondary schools. What I shall write will not be in the nature of a scholarly address in the sense that I will draw upon statistics or quotations from scholarly sources. Instead, I shall present my personal observations, judgments, and, I suspect, prejudices. My justification for the rather generalized nature of what I present is that I do have a genuine belief that progress in any field of specialization in the curriculum is dependent upon a reasonably clear picture of the general trends and orientation of the total educational organization and of the other fields of the curriculum.

It is trite to state that secondary schools have gone through many changes in the last half century, but it is certainly true. What we must of course realize is that this change is not, as we often suppose, something that has happened, but actually is still happening. It is important to try to secure a sense of where we are today and what the future possibilities are because we can by so doing determine at least in part what future changes will be.

Since 1900 we have moved from a high school enrollment of about eleven per cent of the total eligible age population to an enrollment in excess of seventy per cent of the eligible youth, with all of the increased heterogeneity that this involves. In this period, the school has had to attempt to meet an ever increasing variety of educational needs as the complexity of our culture has increased at a phenomenal rate.

A whole host of educational innovations has resulted, often somewhat willy-nilly. The per cent of the high school population enrolled in mathematics courses has decreased; we have developed much more of an elective system; a multitude of new subjects has been added to the curriculum; double-track and even triple-track systems have been developed in some of the academic subjects.

Faced with the diverse interests of students, or even the lack of interest in anything ordinarily related to school, we have become concerned with curriculum offerings related to the interests of students

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and with how to make subjects interesting. Job analysis techniques have been applied in an attempt to select functional subject matter. Faced with the increased problem of preparing students for effective citizenship and with the high school's serving as an integrating agency for a mass of youth of diverse interests and talents, general education, so-called, has received a tremendous emphasis.

All these trends, and many other things, have conspired at times, I am sure, to develop the impression that the major fields of academic learning, such as mathematics, are being relegated to an increasingly minor role. I do not believe that this is essentially true, but certainly with this rapid development there has arisen much misunderstanding. Parenthetically, I would add that I am not one who views the modern secondary school as a deteriorated enterprise or a failure as an educational institution. Far from it. In my judgment, a more than commendable record of adaptation has been made in a period of mass increase in enrollment unlike that ever faced by any other educational institution in history. But inevitably, misunderstandings and disagreements occur, and often dogmatic positions on these areas of disagreement make objective consideration of solutions difficult.

In particular, I would note some of the "either-or" positions that seem to characterize our discussions today. Those who are concerned with the smaller proportion of high school students enrolled in academic courses view current trends with alarm. These persons do not analyze the different school population base upon which comparative figures are drawn or the possible need for adaptation and improvement of instruction in these academic fields. Those convinced of need for wholesale curriculum change apparently espouse the idea that anything that may conceivably be taught in school is as important as anything else. They jockey themselves into defending the importance of the trivial. Those concerned with general education and the adaptation of curriculum organization to this phase of the program, by inference, seem to imply that no part of the curriculum has to deal with the preparatory, vocational, and personal-interest needs of youth. Those with primary interests in special fields are very likely to react with disdain to the whole area of general education. Those concerned with the necessary adaptations of standards to diverse ability groups seem to be arguing that there are no external standards for anybody in any endeavor and, conversely, others concerned about this adopt an equally untenable position of a common standard for all. One final example—perhaps even more sweeping than others cited—is the discussion of whether our schools should be oriented to the average or the gifted student, with the implication that it must be one or the other.

Disagreement is an inevitable and presumably desirable accompaniment of any change, provided that the disagreement does not extend to a conflict that makes objective consideration of a problem difficult. Examples indicate the danger of finding ourselves searching for "either-or" answers, for simple panaceas for complex problems—and I am convinced there are few either-or answers or simple solutions to educational problems. Neither tradition nor change for change's sake offer any answer. Our schools are multipurpose institutions and must continue to serve all youth, to provide general and special education, and to attempt to serve the talented, the average, and those of lesser academic ability. This is a large task but not an impossible one, and the faith that it can be done is the hope upon which we should build.

With these hasty and sketchy observations upon educational developments, I shall now present some observations upon current trends and needs, particularly as they relate to mathematics. Some of this I am certain is very much personal judgment and as such is presented.

I believe we are seeing a renewal of interest in more effective programs for the

able student and the beginnings of some promising developments. While the education of the gifted student is always crucial, the demands of our professions, and particularly the demands in the scientific and research fields, are serving to highlight the need for attention. The potentialities for the progress of the able student beyond what is ordinarily encompassed in the elementary and secondary school program have certainly not been fully explored. I am convinced that there are many able students in high school who could progress in mathematics, if given the opportunity, well into what we have conventionally considered college work. The experimental programs now being conducted in this type of advanced progress seem highly promising to me and preferable to a general acceleration through school and college. I am convinced that if we do not provide these opportunities in our schools that special schools or out-of-school agencies will arise to provide such opportunities.

Personally, I have been associated with the point of view that tended to emphasize the need for general education well through the secondary school. However, as I now view the situation, there are several reservations which I have arrived at in recent years. I doubt the neat decision implying that a field of study serves a single purpose. There is much of a broadening and liberal nature in specialized fields if the instruction has any breadth of direction and purpose. Finally, I doubt the desirability of the delay of specialization to the extent we have at times postulated. From an organizational standpoint, it is very neat to postulate a program of general education until perhaps the completion of the first two years of college, with specialized and professional programs after that period. While perhaps this is appropriate as a matter of degree or emphasis, the nurturing and development of special interests should begin long before that. There are many specialized fields of interests and endeavor which capture the curiosity of children at an early age, while many elements of general education need maturity and often follow, rather than provide, special interests. The identification and stimulation of future mathematicians and scientists, as well as potential specialists in other fields, can and should begin early in the school program.

I am concerned, as I am sure many of you are, that more students do not continue beyond the required minimum in mathematics. I might say that I am not concerned for the statistical reason that a smaller percentage of students in high school are enrolled now than in 1900. I know of no fixed percentage that should be enrolled (and the percentage comparisons are made on two different and noncomparable bases). And I am not convinced that because mathematics is important to our technical culture that everybody must enroll in a four-year program in mathematics in high school. I am concerned because there seems to be a tradition that enrollment in mathematics is solely for those who have a professional need for it. This very attitude, I suspect, tends to discourage even the student with professional interests. I am convinced that there is a substantial pool of students whose general as well as professional education would be enriched by mathematics. There is a real need in the fields of research in the social sciences for those who have a genuine understanding of mathematical concepts. What is required to encourage more students to view mathematics as a logical, rewarding and interesting field for study I must admit I do not know. To hazard a few guesses is perhaps not inappropriate.

One element may be in our counseling and guidance programs and the presentation of better information on courses in mathematics and the contribution of mathematics to many fields of special interest and to many occupations. However, I am inclined to believe that fundamental consideration needs to go further.

We must examine the degree to which mathematics has been made meaningful to

students. I suspect it has been too much taught as a game of manipulation and tricks. The theory of student interest has been grossly misinterpreted when it is assumed that this is developed only by immediate and practical application to personal problems. I doubt if interest in mathematics at any advanced level is sustained by application of problems to a personal budget, measuring the football field, or drawing pretty geometric designs. Real interest arises not necessarily from immediate application, useful as this may incidentally be, as much as it does in beginning to glimpse mathematics as a system of logic and a language with a coherence, structure, and beauty of its own. The study of man's invention of a way of thinking through mathematics is not impossible to communicate to high school and even elementary school students. This type of understanding on the part of the student is what can make mathematics a challenging venture and a meaningful enterprise.

You are faced with a real problem and a real challenge. Mathematics is a live and developing field. I would certainly urge much more daring in experimenting with revision of mathematics content. Real research on how to develop understanding of mathematics as a live and vital system of solving problems and presenting meaning is needed. I am temperamentally inclined to believe the study of mathematics should improve thinking, but I must admit the evidence is far from convincing unless you ask me to accept it on faith and by a definition that is contrary to the spirit of mathematics. I should like to know what the evidence is and whether we are teaching mathematics in a way to produce what we maintain is possible.

The beginnings of an experimental program is underway at the University of Illinois to try out the reorganization of content and the sequence of mathematics in high school. I can personally claim no credit for the development of this experimental project. What the outcome of this experimental project will be is undetermined, as I am certain those who are directly concerned will agree, but the attempt to develop materials that will provide more meaning, produce better integration among fields of mathematics, and introduce new and developing concepts of modern mathematics are more than worth an experimental tryout. I am sure there are other similar developments and much creative thinking about these problems in professional groups. The function of an organization such as the NCTM is to stimulate such thinking rather than to defend the ramparts against all questioning and change.

I should like to make one final observation about the approach to curriculum development. It must be co-operative in a real sense. I happen to be professionally what is termed by some an "educationist." I most certainly deplore the dispersion of effort which seems to assume two different and conflicting points of view between those whose primary interest has been in the so-called subject field and those concerned professionally with the teaching process and the organization of education. As one of the latter group, I would, I suspect, naturally be convinced that there is a contribution to be made by those who deal with the nature of learning and teaching and of human abilities, as well as by those with understanding of the content of subject matter.

As a conclusion, I would observe that most misunderstandings seem to occur when representatives of different groups exchange sweeping generalizations at long range. Contrariwise, I have been encouraged and impressed with the accord and progress that can be made when specific problems are attacked co-operatively. There is a contribution to be made by all concerned. Problems in our world are not solved by pronouncement or by divine judgment. We make progress by the work of many devoted people.

Mathematics as a profession[1]

H. J. ETTLINGER, *University of Texas, Austin, Texas.*

Why teach? You will find an answer in the remarks of an instructor who enjoys teaching.

THE TEACHER'S INFLUENCE

I AM GRATEFUL for this opportunity to address the National Council of Teachers of Mathematics. Let me begin by paying tribute to *my* teachers, especially to those in elementary school and high school, and to those who directed my training through undergraduate and graduate years of study before I became, in turn, a teacher of mathematics in the University of Texas forty-three years ago.

I was a shining example of that not-too-rare college freshman who always boasts that his high school instruction in mathematics was better than that he received as a freshman at college. So you will understand why I take this opportunity to thank my high school mathematics teacher, Miss Cora Butler of St. Louis, Missouri. To her deep interest and stimulation, to her gracious kindness, and to her tireless enthusiasm for mathematics, I owe the inspiration that motivated me in choosing mathematics as my life's profession.

If the subject assigned to me, "Mathematics as a Profession," sounds like the traditional "Apologia Pro Sua Vita," please be compassionate. Bear in mind that I am following in the footsteps of a noble tradition of those who have gone before, the great group of those devoted to an ideal. A span of forty-three years of preoccupation with teaching youth that one and one makes two, calls for self-justification.

Lest you think my point of view is that of reminiscing, I must emphasize that I am looking forward with keen anticipation to many more years of a crowded and busy schedule. I have only begun the work which today seems more important, more rewarding, more attractive than ever. Yes, forty-three years have come and almost gone, and they have been rewarding years, filled mostly with pluses and only a few minuses. I wish to express appreciation and gratitude to the forty-three generations of college freshmen and other students whom I have joyfully taught. I am sure that many of them have painful memories of my class; but I can only hope the years have mellowed their experiences with a hard taskmaster. Unfortunately Dale Carnegie was not my teacher and my own boyhood was full of lessons which develop a fine fighting Irish spirit. You will understand, therefore, why it has been allegedly, but I hope not reliably, reported by the dean of the graduate school that "It is impossible to pass one of his final examinations in a graduate mathematics course without cheating."

But in all seriousness, I want to give heartfelt appreciation for the five or six thousand young people whom I have had. They have always been more than fair and more than co-operative. They have accepted willingly any responsibility placed upon them. Only rarely have they shirked the task which was assigned.

May I cite a letter, received just two weeks ago from a professor of electrical engineering, as an example of one reward of teaching:

[1] An address delivered before the National Council of Teachers of Mathematics, Washington, D. C., December 29, 1955.

I have often thought of what a beneficial effect on my career my studying under your direction has had. The mathematics and the precision of thinking which you presented to me have unquestionably been largely responsible for the particular direction my career has taken. I am, and have been for many years very grateful to you for it.

The attractions of mathematics

You may come to the conclusion that rather than discussing the professional aspects of mathematics I am giving attention to the amateur side of pursuing the vocation of a mathematician, the intangible rewards and appreciations which accrue to a dedicated service in this broad scientific field, to the inner satisfaction and happiness of imparting to the next generation a small part of the treasured heritage of many predecessors and the rich body of knowledge whose history reaches far back. However heartening this may be, what of a future in the profession of mathematics from a worldly point of view? I sometimes feel that many teachers would gladly exchange the appreciation and warm regard of students for something more substantial in financial return. At the present, devotees of mathematics are operating in the seller's market and it is high time that the competition of business and industry produce results in higher salaries for teachers of mathematics. If the White House Conference on Education is any sound indication of the level of public opinion, much may be hoped for from this forward look.

The subject of mathematics is as vast as the universe and as venerable as time itself. But I am not to speak about mathematics, or even about the applications of mathematics, but primarily about those who profess mathematics.

What type of person professes mathematics? In his book *Men of Mathematics*, E. T. Bell deals with the private lives of mathematicians, and in two cases includes women. Personalities like Thales of Miletus, Pythagoras, and Archimedes among the ancients are fairly definitely delineated and even pictured, though somewhat in fanciful fashion. Such medieval men as Cardan and Tartaglia, Wallis and Lord Napier who preceded the recognized invention of the calculus by Leibnitz and Newton, down to the moderns, Cauchy and Gauss are described. Most of these men were teachers, though there was an occasional lawyer or physician, and more rarely, a gambler. Bell emphasizes the human interest stories of these famous mathematicians, such as death in a duel, a bitter feud between contemporaries over priority of discovery and possible plagiarism, and the love life of a famous woman mathematician. These are spicy elements which would hardly adorn an elementary school course and would probably be used by only a daring college instructor. It is quite in order, however, to point out that on the whole the lives of most professional mathematicians are not too different from their fellow townsmen in other walks of life.

Teaching and doing mathematics

There is a feeling abroad in the market place, in the community, and even on the campus, that "doing" or creating mathematics and teaching mathematics never go together. A young instructor may be told that he must choose to be a fruitful research worker in mathematics or a successful teacher; he is told that it is futile to try to be both. It was recently my privilege to hear a man who is both a public school superintendent and a mathematics teacher address a Texas Council of Mathematics group on "Mathematics and Thinking." His theme was that a high school or an elementary school teacher of mathematics has a modest but important opportunity to "teach and do" mathematics. He pleaded for the encouragement to try new methods of proofs, and the statement of new problems. Within the classroom it is important to present a challenge to the pupil. The routine of daily labor in arithmetic and in algebra, in geometry and trigonometry, presents many opportunities for doing, for thinking, for creating.

The assumption that we have any

responsibility toward our own classroom pupils to make their work enjoyable, meaningful and fruitful, will hardly arouse any bitter discussion in this group. I wish to turn to our obligation to those who are not in our classrooms. Dr. Vannevar Bush, now retiring as head of the Carnegie Institute, has stated that we realize only about fifty per cent of the potential human resources available in our youthful population. There are many reasons for this situation. I have waged an extensive campaign at the grass roots level of the school and community to place the responsibility on the pupils, on their parents, and on the citizens who are immediately and directly responsible for our public schools. I am astonished at the impact resulting from a direct-approach program of fairly short duration. It is producing measurable results.

At the same time, it must be realized that the major part of the task of attracting those outside who should be inside rests with the teachers. A mother in a small community said, "My boy is taking mathematics and chemistry in our high school, but unfortunately, he knows more about these subjects than his teacher. I think the answer lies in more teachers, well trained and qualified." There is no doubt that success in realizing a higher percentage of yield in human resources depends upon more and better teachers. This will bring into our mathematics classes more pupils who have the requisite talents.

The use of various types of aids to teaching is deserving of attention. Without going overboard and characterizing the use of diagrams to represent number ideas and number relations as the greatest invention of the human mind, we can agree with that Chinese proverb, "One picture equals fifty thousand words." This is indeed helpful, but it should be remembered that one basic idea completely and well presented and taught will equal fifty thousand special devices. The appreciation and understanding of a fundamental principle of mathematics is more than equal many thousand fold the results which follow from the use of gadgets.

Business and industry

Many agencies and groups are deeply interested in helping make the profession of mathematics financially rewarding. Our own National Council of Teachers of Mathematics is of great value professionally through its publications, *The Arithmetic Teacher* and The Mathematics Teacher. These are a genuine source of inspiration. Business and industry are deeply interested. I learned of this at first hand when I was invited to attend a meeting of the NTCM Committee on the Co-ordination of Mathematics with Business and Industry. The representatives of business and industry are not only interested in mathematicians who serve their needs, they are vitally concerned about teachers of mathematics. They realize that industry's recruitment of trained minds has contributed to the critical shortage of mathematics and science teachers. Consequently, we have the present large scale participation by business and industry in scholarship and fellowship programs and invitations to accept summer employment in offices and laboratories.

Business and industry are ready to participate in many other ways to raise standards of achievement in the teaching of mathematics, just as they have set up internal educational programs for training, which draw on their staff for teaching personnel. Business and industry can help create the climate of public opinion which will crystallize the general interest manifested at the White House Conference on Education into definite recognition that results in heightened status through increased income. The late Professor G. A. Miller of Illinois is a very rare example of one mathematician who by wise investment of a small salary and the use of group theory and the exponential law of compound interest was able to multiply his dollars into nearly a million. The rank and

file of mathematics teachers are, however, dependent on improvements of salary scale to secure recognition in substantial form.

There is another aspect of the improvement in the status of mathematics as a profession commensurate with its importance in our national life and community life. Citations of recognition at the community level on the occasion of a significant school and community program would be helpful and might well be accompanied by a cash prize. Industrial capital and foundation grants devoted to this purpose would bring large dividends in community and national welfare.

Noblesse oblige

You may recall that the followers of Pythagoras regarded themselves as spiritual kin, as members of a religious brotherhood. Many dreamed of a Golden Age of understanding. Those who today profess mathematics can form a dedicated and consecrated group of devotees. Just the other day, one of my students, an alert, intelligent, and personable young lady who plans to teach mathematics, related how her roommate challenged her by asking, "Why are you studying so much mathematics? You will probably soon be a housewife." For both of these young ladies I want to quote Dr. William Vermillion Houston, President of Rice Institute, in an address entitled "A Philosophy for the Twentieth Century."

Said Dr. Houston, "The importance of science is not based on the material or military benefits it may bestow. I would urge the study of science in our schools as a means of developing a philosophy of life, and would suggest that it needs most to be studied by those who do not plan to be scientists. It must be studied most by those concerned with human relations, with matters of the spirit."

If you remember that mathematics is an important part of science, then professors of mathematics and those who teach it, can feel the inspiriting influence of devotion to a field of study which will replace credulity and superstition with conviction and faith in the foundations of our way of life, not as a replacement for our religious beliefs and spiritual ideals, but as a strong substructure to those ideals which enrich all aspects of our living. This could well be our tenet, a tie to bind all of us teachers of mathematics in the genuine feeling that we are members of a noble profession.

A DATE TO REMEMBER

Thirty-Fifth Annual Convention

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Bellevue-Stratford Hotel
Philadelphia, Pennsylvania
March 27–30, 1957

What is ratio?

CHARLES BRUMFIEL, *Ball State Teachers College, Muncie, Indiana.*

Key mathematical concepts need some careful thought. This article examines the concept of ratio and relates it to the quotient of two numbers.

MOST HIGH SCHOOL ALGEBRA TEXTS do not give an acceptable definition of a *ratio*. In contrast, the better college algebra texts usually do so. The high school text may say that a *ratio* is *a comparison of two quantities expressed as a fraction*. This is completely unacceptable. It seems to say two things:

1. A ratio is a "comparison of two quantities."
2. We indicate a comparison of two quantities by writing a "fraction."

One might properly ask now for a definition of this strange thing, "a comparison of two quantities." It would also seem reasonable to inquire whether or not the assertion that we *express* this comparison as a fraction means that a comparison *is* a fraction. On the other hand, the college text will usually give the brief and acceptable definition that the *ratio of number* a *to number* b *is the number we get by dividing* a *by* b. If a and $b \neq 0$ are fractions, then their ratio is a fraction. But this ratio may well be an irrational number rather than a fraction, if a and b are not fractions.

To understand why we have persisted in repeating such a ridiculous "definition" down through the centuries we must go back to the introduction of the word into mathematics. The Pythagorean school of mathematicians was apparently the first to speak of the *ratio of line segment* AB *to line segment* CD. This implied that a unit segment existed which would *measure* both *AB* and *CD*. In modern terminology we would say that when *AB* and *CD* were measured by this unit segment, their lengths were integers. This concept formed the basis for the development of many theorems on similarity.

When it was learned that pairs of incommensurable segments existed, a logical scandal erupted. The validity of the proofs of theorems on similarity was destroyed. It is well known how the great Greek geometer Eudoxus restored rigor to this section of geometry by formulating a beautiful definition of "equality of two ratios," i.e., *proportion*. Euclid copies the work of Eudoxus in his *Elements*. Let us recall this treatment. Euclid calls a ratio "*a sort of relation with respect to size between two magnitudes of the same kind.*" He asserts that a pair of magnitudes does have this relation called *ratio* subsisting between them if an integral multiple of each one can be found which exceeds the other. To the Greek geometers, then, a ratio was a *relation* which held between certain pairs of geometric figures, say pairs of segments, angles or polygons. A ratio was not a *relation* between two numbers.

There were good reasons for the above point of view. The Greeks had no real number system. It was impossible for them to attach *numbers* to segments, angles, and polygons to represent lengths of segments, measures of angles and areas of polygons. The Greeks were compelled to speak as they did when they dealt with concepts of ratio and proportion because there was no measurement in Greek geometry. Geometric configurations had to be compared with-

out measurement. The creation of the real number system and the invention of *analytic geometry*, made possible by the assumption that the set of real numbers can be placed in one-to-one correspondence with the set of points on a line, has of course revolutionized techniques in geometry. Today if we wish to *compare* two segments we can compare their lengths. We do this by dividing the length of one segment by the length of the other. This quotient is our *ratio*. The Greeks could not do this. Their segments had no lengths. Words like *length* and *area* do not occur in Greek geometry, for these things are real numbers, and the number system of the Greeks did not meet the demands of their geometry. It is a most interesting phenomenon that today our texts still use the archaic terminology of 2500 years ago.

A modern mathematician, formulating a precise, abstract definition would perhaps define a ratio as *an equivalence class of ordered pairs* (r_1, r_2) *of real numbers with* $r_2 \neq 0$. We call (r_1, r_2) and (s_1, s_2) equivalent if and only if $r_1 s_2 = r_2 s_1$. We should say, then, that the pairs (1, 2), (4, 8), $(\sqrt{2}, \sqrt{8})$, $(\pi, 2\pi)$, $\cdots$, belong to the same ratio, i.e., are members of the same set, or equivalence class. And now the statement that the *ratio of* 2 to 5 is the same as the *ratio of* 4 to 10 means simply that these pairs (2, 5) and (4, 10) *belong* to the *same ratio*. This is a beautiful definition and certainly appeals to mathematicians. However, for high school mathematics it is undoubtedly too abstract.

Fortunately the definition of the ratio of r_1 to r_2 as the quotient, $r_1 \div r_2$, is equivalent to the modern definition described above. To see this, consider that in our usage of ratio we are interested only in deciding whether, for two given ratios, R_1 and R_2, we have $R_1 = R_2$, $R_1 < R_2$ or $R_1 > R_2$. If each ratio is a set we must formulate *definitions* of "$<$" and "$>$." This is done in the following way. Let (r_1, t_1) be any ordered pair of real numbers in R_1 and (r_2, t_2) any ordered pair in R_2. Then we say $R_1 = R_2$, $R_1 > R_2$, or $R_1 < R_2$ according as $r_1 t_2 = t_1 r_2$, $r_1 t_2 > t_1 r_2$, or $r_1 t_2 < t_1 r_2$. Now this condition is identical with $r_1/t_1 = r_2/t_2$, $r_1/t_1 > r_2/t_2$, or $r_1/t_1 < r_2/t_2$. Whether we call our ratios *sets of ordered pairs of real numbers* or *quotients of real numbers* we have equivalent definitions. In the language of abstract algebra, an *isomorphism* can be established between the two concepts so that it is completely immaterial which one we use.

Since the definition of the ratio of number a to number b as the number $a \div b$, is both mathematically sound and easily understood, it would seem well to use it in place of the vague terminology now in vogue.

Letters to the editor

Dear Sir,

In connection with the article, "The National Status of Mathematics Contests," by Daniel B. Lloyd, in the October 1956 issue of THE MATHEMATICS TEACHER, we, the mathematics teachers of East Texas, would like to have you know about our mathematics meet that we hold in East Texas in a different city each year.

We feel that this meet has created more interest in math than anything else that we have ever done. The only fault that we can find is that it has grown so large that we have difficulty with housing. It is an invitational meet to schools in East Texas, with schools entering as many participants as they desire.

We think that it is the only meet of its kind in Texas, maybe in the nation.

The Meet will be held in Longview in March, 1957. I am enclosing a copy of the meet that was held in Kilgore last year with some 600 participants.

Yours truly,
C. N. Wilkinson
Mathematics Instructor
Longview Senior High School
Longview, Texas

Variation—a process of discovery in geometry

CLARENCE H. HEINKE, *Capital University, Columbus, Ohio.*

Variation as a process whereby interesting geometric theorems are suggested is explored in this paper and its advantages noted.

TYPICAL OF ONE TYPE of inquiry frequently practiced by children is the question asked by a second grade girl, "Mother gets new African Violet plants by breaking off little shoots and putting them in water until they grow roots. *What if* we try the same thing with roses? Can we get new rose plants in the same way?" Much adult activity, including that of mathematicians, which is appraised as "original" or "creative" also seems to result from asking specific questions of the sort, "*What* happens in a given situation with which I am familiar *if* certain changes are made?"

Each system of non-Euclidean geometry, for example, may be viewed as the result of deliberately making certain changes in the set of assumptions on which the geometry of Euclid is based and then seeking an answer to the question, "What theorems follow from this new set of assumptions?" It seems altogether reasonable, moreover, to assume that the early geometers, who discovered most of the theorems studied by high school pupils of the present day, frequently asked the question, "*What if . . .?*"

Examination of the content of elementary plane geometry reveals that many of its propcsitions can be discovered by applying the question "*What if . . . ?*" to other propositions already known. It is the purpose of this article to report some of the results disclosed by a detailed study of this matter.[1] It is hoped, further, that other teachers may be encouraged to try some of the procedures suggested.

SOME EXAMPLES

There are numerous examples of geometric statements[2] commonly found in textbooks that are the same except for one or a small number of salient words. Some such illustrations follow.

Illustration 1

Statement 1.1 The midpoints of the sides of a *parallelogram*, taken in order, are the vertices of a parallelogram.

Statement 1.2 The midpoints of the sides of a *trapezoid*, taken in order, are the vertices of a parallelogram.

Statement 1.2 is identical with Statement 1.1 except that the word *trapezoid* replaces the word *parallelogram*. After Statement 1.1 is known, Statement 1.2 can be discovered upon asking and pursuing the question, "What is the form of the figure obtained by connecting the midpoints of the sides of a quadrilateral

[1] Clarence H. Heinke, "Discovery in Geometry Through the Process of Variation," (Unpublished Ph.D. Dissertation, Ohio State University, 1953).

[2] *Statement*, as used here, means "any group of words, or a group of words together with other symbols, which expresses something that is either true or false." Thus theorems, corollaries, and exercises, as these words are used in geometry, are all connoted by the term *statement*. Moreover, a statement may consist of words only, or it may be expressed by a figure and associated itemized data and conclusion(s).

which has only one pair of opposite sides parallel?"

If these statements are portrayed in terms of *figure*, *data*, and *conclusion*, the relation between them is seen in another way; Statement 1.2 is the same as Statement 1.1 except for the deletion of a datum.[3]

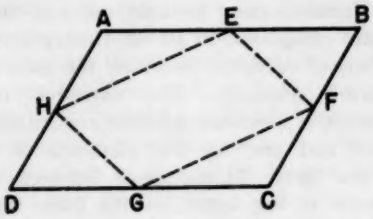


Figure 1

Statement 1.1

Data:

(a) $ABCD$ is a quadrilateral
(b) AB is parallel to DC
(c) AD is parallel to BC
(d) E, F, G, and H are the midpoints, respectively, of AB, BC, CD, and DA.

Conclusion: (x) $EFGH$ is a parallelogram.

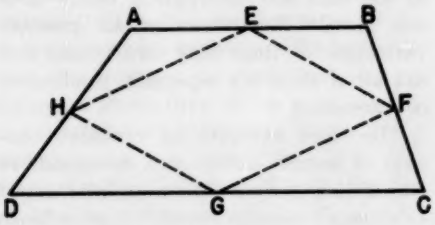


Figure 2

Statement 1.2

Data:

(a) $ABCD$ is a quadrilateral
(b) AB is parallel to DC
(c) (Deleted)
(d) E, F, G, and H are the midpoints, respectively, of AB, BC, CD, and DA.

Conclusion: (x) $EFGH$ is a parallelogram.

[3] *Data*, is used instead of, but with the same meaning as, the more commonly used words *hypothesis* or *given*.

Inspection of the itemized forms shows that Statement 1.2 differs from Statement 1.1 only in the deletion of one datum; the conclusions are the same. Of course, this sort of circumstance is unusual. More frequently, when a datum is deleted to form a new statement, the conclusion must be changed also. *Illustration 2* portrays this kind of situation.

Illustration 2

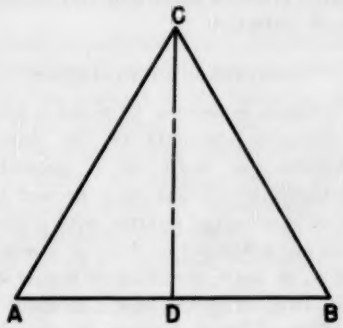


Figure 3

Statement 2.1 The bisector of the angle *between the equal sides of an isosceles triangle bisects the base.*

Data:

(a) Triangle ABC
(b) AC is equal to BC
(c) CD bisects angle ACB.

Conclusion: (x) $AD = DB$.

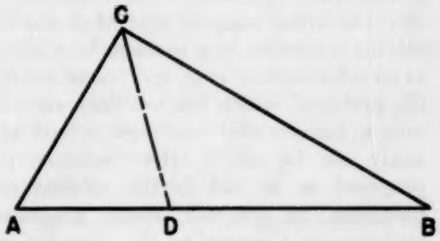


Figure 4

Statement 2.2 The bisector of an interior angle of *any* triangle *divides the opposite side into segments which are proportional to the sides adjacent to the angle bisected.*

Data:

(a) Triangle ABC

(b) (Deleted)

(c) CD bisects angle ACB.

Conclusion: (x′) $AD/DB = AC/BC$.

With deletion of the datum that two sides of the given triangle are equal, it becomes necessary to modify the conclusion also. Consideration of just how the new conclusion may be discovered is deferred pending definition and amplification of variation.

Definition of variation

Variation is defined to mean a process of changing elements of the data, or conclusion, or both, of a geometrical statement, which has been proved to be true or is accepted as true, with a view to obtaining a *new* set of *data*, or a *new conclusion*, or both, resulting in a new statement. Ordinarily the new statement is to be affirmed by deduction or else shown to be false; on some occasions it may be adopted as a new assumption. As employed here, the word *variation* has a meaning so different from its other use in mathematics to denote inverse, direct, and joint relations between variables that no confusion is anticipated.

In fact, the present use of the word *variation* is not without precedent. In Polya's book *How To Solve It* there is a section titled "Variation of the Problem."[4] In it the writer suggests that when one is seeking a solution to a problem he might, as an intermediary step, try "variation of the problem" which has not been solved into a form which has been solved or easily can be solved. Here *variation* is proposed as an aid in the *solution* of problems. In contrast, Bakst proposes *variation* as a means to the *formulation* of new problems; he says,

> The greatest advantage which may be derived from the study of geometry is associated with the opportunity of variation of certain given conditions stated in a problem, whether this is an original or a construction. If the pupils are encouraged to vary the conditions, they will eventually develop the tendency to "imagine" and to formulate problems of their own.[5]

It is in the general spirit of this latter comment that the word *variation* is used in this paper.

Variation may be analyzed into three main categories. In the first category are kinds of variation in which the data are varied originally. The variation may consist of deletion, addition, or substitution and may involve elements of the given figure, relationships between elements of the figure, or the basic figure itself. Moreover, the variation may involve numerical data, data not expressed in numerical terms, or both. Variation of the data is then followed by a determination of what conclusions can be inferred and deduced. In the second kind of variation, changes are made initially in the conclusion. These are similar to the ones that may be performed on the data. Following the changes in the conclusion, those that can or need to be made in the data are determined. Space does not permit illustration of all possible variations of these first two types, and not all of them are especially productive or interesting.

The third category of variation consists of certain well-known operations in which data and conclusion are rearranged or changed simultaneously in accordance with prescribed procedures. This category includes the principles of duality and reciprocity,[6] and the formation of converses, inverses, and contrapositives.[7] A

[4] G. Polya, *How To Solve It*. (Princeton, New Jersey: Princeton University Press, 1945), pp. 182–86.

[5] Aaron Bakst, "Mathematical Recreations," The Mathematics Teacher, XLIV (April, 1951), 270.

[6] An excellent reference is David Eugene Smith, *The Teaching of Elementary Mathematics*. (New York: The Macmillan Company, 1900), pp. 275–77.

[7] Nathan Lazar, *The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry*. (New York: Copyright by the author, 1938). Also published in The Mathematics Teacher, XXXI (March, April, May, 1938), 99–113, 156–74, 216–40.

comment concerning variation and inverses follows in a later section.

Discovering either the conclusion or the data after changes have been made in the other

In Illustration 1 the data for Statement 1.2 is obtained by deliberately deleting a datum from Statement 1.1. It happens that the conclusion is the same after deleting this datum as before, but how can this be discovered? A first question might be, "Does the original conclusion remain true even after a datum has been deleted; that is, is a parallelogram obtained when the midpoints of the sides of a trapezoid are joined, as was the case when the midpoints of the sides of a parallelogram were connected?" A procedure for following up the question is to draw several trapezoids of different shapes and sizes, connect the midpoints of consecutive sides, and inspect the resulting figures. Of course, all the figures obtained will appear to be parallelograms; the next step, then, is to attempt to prove this deductively.

However, more frequently than not, changes in the data are accompanied by changes in the conclusion. The hopelessness, for instance, in Statement 2.2, of trying to prove that AD is equal to DB is apparent from a figure in which AC is much shorter (or longer) than BC. The conclusion that is indicated might be discovered by making measurements of AD, DB, AC, BC, and perhaps other parts in triangle ABC, and then noticing from this information that AD/DB appears equal to AC/BC. However, the correct conclusion might also be discovered by observations and reflections somewhat like the following: AD does not seem equal to DB if AC is not equal to BC; but AD is less than DB if AC is less than BC; hence, two relations, *both of which are true in the original statement*, appear possible, (1) $BC-AC=BD-AD$, and (2) $AD/DB=AC/BC$. Only the latter relation can be substantiated by measurement and by proof. Of course, considerable experience and insight are required even to think of relation (2).

It is true, in general, that experience and insight, gradually developed through continued utilization of the process of variation, help disclose what initial changes can be made advantageously and what other changes, if any, should accompany the initial changes.

Further illustrations of variation

In both *Illustration* 1 and *Illustration* 2 variation is performed by initially deleting one of the original data. Variation may also be accomplished by substituting one datum for another, as is shown in *Illustration* 3, below. An exercise found in the introductory work in some geometry books is the point of departure. The datum that is varied is expressed numerically.

Illustration 3

Original Statement 3.[8] The bisectors of two adjacent supplementary angles are perpendicular.

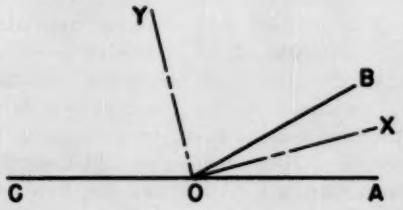


Figure 5

Data:
(a) $\angle AOB+\angle BOC=180°$
(b) XO bisects $\angle AOB$.
(c) YO bisects $\angle BOC$.
Conclusion: (x) $\angle XOY=90°$

Conscious seeking for opportunities to employ variation might suggest several questions. One such question is, "What

[8] Hereafter the statement from which one or more new statements are obtained is called the *original statement;* the new statements, whether expressed as word sentences or in data, conclusion and figure form, are called *variants.* Variants are numbered serially in each illustration.

is the angle between the bisectors of two adjacent angles whose sum is 90° instead of 180°?" Thus the data for variant 3.1 is obtained.

Variant 3.1

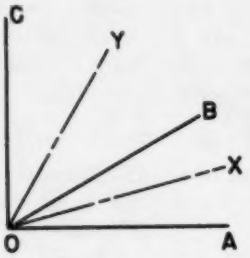


Figure 6

Data:

(a′)[9] $\angle AOB + \angle BOC = 90°$

(b) XO bisects $\angle AOB$

(c) YO bisects $\angle BOC$.

By measurement, by building a proof analogous to that for original Statement 3, or by "just seeing it," the conclusion which follows is conjectured.

Conclusion: (x′) $\angle XOY = 45°$.

A conclusion is, of course, tentative until established by deductive proof or until the data and conclusion together are adopted as an assumption. After proof or decision to adopt as an assumption, a word statement should generally be formulated. (All proofs are, however, omitted from this paper.)

Word Statement 3.1. The angle formed by the bisectors of two adjacent complementary angles is 45°.

An endless number of similar variants can be formed, each with the sum of the adjacent angles specified as some numerical value, but not exceeding 180°. After few or many such variants, depending on the insight of the individual, an expected observation is the one expressed as Variant 3.2.

Variant 3.2. The angle between the bisectors of two adjacent angles whose sum does not exceed 180° is equal to one-half the sum of the given angles.

This variant may be looked upon as being obtained by replacing a datum expressed in numerical terms by one not expressed numerically. It may also be viewed as a generalization of all of the preceding variants.

Illustration 4

Familiar concurrency theorems concerning particular lines associated with a triangle, the medians, the angle bisectors, the altitudes, and the perpendicular bisectors of the sides, can be obtained as variants of any one of them after that one has been discovered or is known in some other way. Suppose the first one known is, "The *medians* of a triangle are concurrent." Then the remaining theorems are obtained by substituting *angle bisectors*, *altitudes*, and *perpendicular bisectors of the sides* for *medians*. In terms of the analysis presented earlier, this illustrates substitution involving elements of the given figure.

Illustration 5

The individual theorems concerning the measurement of angles in terms of arcs intercepted on a circle may all be derived as variants of the basic statement about the central angle. One possible sequence is presented in this illustration.[10] Moreover, the theorems can be proved in the order presented.

Original Statement 5. A central angle of a circle is measured by its intercepted arc.

Variant 5.1. The angle between a diameter and another chord intersecting within a circle is measured by one-half the sum of the intercepted arcs.

[9] Successive primes applied to an item of data or conclusion designate successive variations of that item.

[10] For another interesting sequence see "Outline of Geometry," *Report of the School Committee of the City of Boston*, (School Document No. 8, 1934) p. 34.

Variant 5.2. The angle between any two chords intersecting within a circle is measured by one-half the sum of the intercepted arcs.

Variant 5.3. The angle between two chords intersecting on a circle is measured by one-half the intercepted arc.

Variant 5.4. The angle between a tangent to a circle and a chord passing through the point of tangency is measured by one-half the intercepted arc.

Variant 5.5. The angle between two secants, two tangents, or a secant and a tangent intersecting outside a circle is measured by one-half the difference of the intercepted arcs.

Each variant is produced by substituting in the data of the preceding statement, adopting a tentative conclusion in one of the ways previously described, establishing a correct conclusion by deductive proof, and finally formulating the corresponding word statement.

Illustration 6

Illustration 1 deals with the figures obtained by connecting the midpoints of the sides of particular quadrilaterals. There are as many such statements as there are types of quadrilaterals, and the midpoint figures are not all alike. A possible point of departure for discovering all these statements might not involve quadrilaterals at all, but triangles. Original Statement 6, which follows, is found in many geometry books in the work on tri[illegible]gles.

Original Statement 6. The figure formed by connecting the midpoints of the sides of a triangle is another triangle similar to the given triangle.

Expected to follow, now, is the "what-if" question, "*What* sort of figure is obtained *if* the midpoints of the sides of a quadrilateral are connected?" (Note that the data substitution involves the basic figure.) Although the figures that may be drawn suggest that the resulting figure appears to be a parallelogram, proving this is not easy. Appropriate strategy is to add to the data. The given quadrilaterals then become special types, and the midpoint figures are special parallelograms in some cases. For example, if the given quadrilateral is an isosceles trapezoid or a rectangle, the midpoint figure is a rhombus, and if the given quadrilateral is a rhombus the midpoint figure is a rectangle. Some of the statements thus obtained are proved quite easily, and all should be formulated and investigated. Experience gained with these special cases provides help in devising a proof when the given figure is a general quadrilateral.

The reader is familiar with some of the statements referred to and can easily investigate the others. Three variants, involving the conclusion initially, are perhaps not so well known. For the sake of discussion, assume the following variant to have been proved.

Variant 6.1. The midpoints of the sides of a *quadrilateral*, taken in order, are the vertices of a *parallelogram*.[11]

A fruitful question at this point is, "*If* the midpoint figure of a quadrilateral is to be a rectangle, *what* properties must the given quadrilateral possess?" This question implies an addition to the conclusion of Variant 6.1, namely, that one angle of the parallelogram be a right angle. The required property is suggested by the statement, assumed to have been discussed earlier, "The midpoints of the sides of a rhombus, taken in order, are the vertices of a rectangle." A property possessed by the rhombus, but by a larger class of quadrilaterals also, and the one desired as a datum in this instance, is that the diagonals be perpendicular. Following proof, Variant 6.2 can be stated.

[11] It is commonly understood that convex quadrilateral is meant when the word *quadrilateral* is used without type specified. The statement is true also, however, for concave and even "crossed" quadrilaterals.

Variant 6.2. The midpoints of the sides of a *quadrilateral with perpendicular diagonals* are the vertices of a *rectangle.*

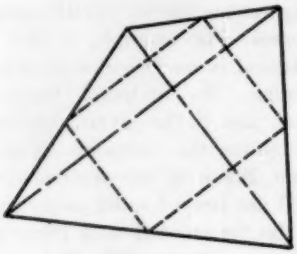


Figure 7

Two additional variants follow which can be formed by initially substituting in the conclusion, or adding to it, and then determining the needed data.

Variant 6.3. The midpoints of the sides of a *quadrilateral with equal diagonals* are the vertices of a *rhombus.*

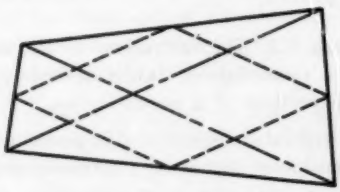
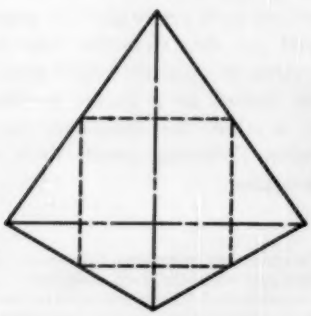


Figure 8

Variant 6.4 The midpoints of the sides of a *quadrilateral with diagonals both equal and perpendicular* are the vertices of a *square.*

Figure 9



Illustrations have been given which show variation of the data initially and of the conclusion initially. As stated previously, certain prescribed manipulations such as use of the principle of duality and the formation of converses, inverses, and contrapositives may also be viewed as variations. Consideration of these is, however, omitted, except for a comment about the inverse.

Variation and the Inverse of a Statement

With the explanation, "every theorem that contains many conclusions can always be subdivided into as many theorems as there are conclusions, all having the same hypotheses,"[12] the following definition has been given for *inverse:* "An inverse of a proposition having one conclusion may be formed by contradicting one of the hypotheses and the conclusion."[13]

An important contrast between variation and the formation of inverses is implied in some of the preceding illustrations but has not been enunciated. An illustration will help clarify and emphasize this contrast. Consider one of the statements indicated in *Illustration* 6, "If the midpoints of the sides of a rhombus are connected in order, the resulting figure is a rectangle." An inverse is: "If the midpoints of the sides of a quadrilateral which is not a rhombus are connected in order, the resulting figure is not a rectangle."

This inverse is manifestly not a true statement, as can be seen from the figure provided with Variant 6.2. Although this knowledge might be of value, there are two other considerations of greater importance. (1) If the given quadrilateral is not a rhombus, what other kind of quadrilateral (not already investigated) might it be fruitful to consider? (2) Under the condition of the given quadrilateral being some particular (or general) quadrilateral other than a rhombus, if the mid-

[12] Lazar, *op. cit.*, p. 27.
[13] *Ibid.*, p. 25.

point figure is not a rectangle, precisely what is it? One of the several possible variants is, "If the midpoints of the sides of a parallelogram are connected in order, the resulting figure is a parallelogram." This variant is a positive, direct statement asserting, not something which is not true, but something that is true.

An inverse of Statement 2.1 is "The bisector of the angle between unequal sides of a triangle divides the third side into segments which are not equal." This inverse is a true statement, but it is not nearly as useful as the variant, Statement 2.2, which states that the segments into which the third side is divided are proportional to the other two sides of the triangle.

The occurrence is also found in Euclid of two propositions related as those in the illustrations just given. Proposition 4, Book I, is:

> If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal.[14]

Proposition 24, Book I, is:

> If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides equal to them greater than the angle contained by the two sides equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.[15]

An inverse of I,4, of which I,24 may be considered the "positive form," is: "If two sides of a triangle are equal to two sides of another triangle, but the included angles are unequal, then the third sides are unequal." However, this statement is not found in Euclid, nor is it mentioned by the commentator, Heath.[16]

In short, then, an inverse is obtained by negating the conclusion and a datum in a given statement. The more inclusive process, variation, includes substituting a new datum or data and a new conclusion or conclusions for the negated items. A variant states what is true in view of certain data instead of what is not true, and hence frequently is more useful. This is not to say, by any means, that teaching and learning about the inverse is of little value; its importance has been properly stressed elsewhere.[17]

Variation and generalization

That variation is equivalent to generalization in some instances and fosters it or is an intermediate step to generalization in other instances has probably been recognized by the reader. A satisfactory treatment of the relation between variation and generalization is, regrettably, too lengthy to be made part of this paper.

Values to be expected from using variation

Variation is useful in two distinct ways: (1) The teacher may utilize it as an effective process for developing some of the subject matter recognized as belonging in any course in geometry. (2) He may also, through careful suggestion, guide the students to discover certain specific theorems; by general encouragement he may provide motivation for the students to discover statements which are new to the teacher as well as to the student and which are not commonly found in textbooks. Among the values that may then be expected are the following:

(1) Variation fosters security at each stage of learning where it is employed, for emphasis is on beginning with statements that are known and then changing them a little at a time.

(2) Use of variation affords means of providing for individual differences of ability, time, insight, and interest. Nearly every student capable of grasping what is presented by the teacher can learn to

[14] I. Todhunter, *The Elements of Euclid* (New York: Macmillan and Company, 1896), p. 9.

[15] *Ibid.*, p. 26.

[16] T. L. Heath, *The Thirteen Books of Euclid's Elements.* (Cambridge; The University Press, 1908). I.

[17] See Lazar, *op. cit.*

discover some variants himself. On the other hand, no assignment can ever be considered finished.

(3) Mathematics is sometimes criticized because for every problem proposed to the student there is a solution; thus mathematics provides training in reasoning, but not in making judgments concerning the necessity or adequacy of certain data. Stated with specific reference to plane geometry the criticism is as follows: The student is not asked to prove anything which cannot be proved, and impossible constructions are not proposed; hence, in elementary geometry the student is not required to judge a proposed conclusion as possibly true or possibly not true, and thus he is not challenged to reject a conclusion as untenable. Since it is not the nature of problems as they are met in life situations to have enough and just enough data available, and to be always capable of solution, mathematics fails to provide complete experience with problems.[18] This condition is remedied when the pupil actually discovers and develops mathematical notions himself. In particular, when a student generates new statements by the process of variation he has extensive experience in deciding whether to try to prove or disprove a statement formulated tentatively, and in rejecting it if it is found to be false or changing it further so that it is judged more likely to be true.

(4) One of the values commonly accorded the study of mathematics is the development of habits of precise expression.[19] Exactly how this aim is to be achieved is, however, seldom stated. Frequently when the process of variation is used, a new statement is obtained by just making changes in a figure or in the data and conclusion stated in terms of a figure. Formulating a corresponding word statement after changes have been made in this manner provides very significant experience in the art of precise expression.

(5) Certain results accrue in any classroom that throbs with the excitement of discovery. Since variation promotes discovery, these results may be expected where variation is emphasized. Among them are increased interest, mutual sharing of discoveries, more learning, and improved retention.

(6) Properly presented in the medium of geometry, variation may be expected to be utilized in fields of mathematics other than geometry and in areas of study other than mathematics.

Implications for Teaching

Variation is a *process*, a process performed in a conscious, deliberate manner. As such, the process itself must be emphasized, not only its products. Variation, then, should be introduced early in the study of geometry. The teacher should use it frequently in presenting new material to the class and give specific hints of opportunities for the students to use it themselves. He should encourage students to initiate variations in situations not suggested by the teacher. Particular types of variation should be distinguished and identified when they occur; the students should participate in this process. The principal challenge, both to teacher and students, is to search for familiar knowledge which can suggest a point of departure for presenting or discovering more difficult, but important knowledge.

If the students are to be expected to apply variation in areas outside geometry they must be shown some examples and encouraged to develop others. One of the fine achievements possible in the study of geometry is a renewal of the childhood disposition to initiate fruitful inquiry with the question, "*What if . . . ?*"

[18] Illustrative of this point of view is Dewey's conception of the role of mathematics in relation to the complete act of thought. The contribution of mathematics to the development of the ability to do reflective thinking of a high order is limited largely to providing experiences in one phase of the total process, namely, "reasoning (in the narrower sense)." John Dewey, *How We Think* (New York: D. C. Heath and Company, 1933), pp. 111–13.

[19] See, for example, J. Shibli, *Recent Developments in the Teaching of Geometry*. (State College, Pennsylvania: Published by the author, 1932), pp. 214–15.

Mathematics in the college general education program

T. M. SIMPSON, *Memphis, Tennessee.*

Like general mathematics in the high schools, mathematics for general education in the colleges is not a well-defined subject.

IN "The Poor Scholar's Soliloquy," by Dr. S. M. Corey,[1] the boy says, "I don't do very well in school in arithmetic either. Seems I just can't keep my mind on the problems. We had one the other day like this:

> If a 57 foot telephone pole falls across a cement highway so that 17-1/6 feet extend from one side and 14-9/17 feet from the other, how wide is the highway?

That seemed to me like an awfully silly way to get the width of a highway. I didn't even try to answer it because it didn't say whether the pole had fallen straight across or not." The low estimate into which mathematics has fallen is not surprising when our schools raise up a citizenry with memories of silly problems.

In algebra we have poor old *A* taking 6 days to do a job *B* does in 4 days and we ask how long will they take working together. Or we have the pipe that fills a cistern in 5 minutes less than it takes a second pipe, and so on. Teachers of algebra are familiar with absurd problems like the following: "If the price of eggs rises 15 cents a dozen, one will be able to get 1½ dozen fewer eggs with $6.25 than before. What was the lower price?" No one is likely to figure out the price this way.

As new texts continue to appear, so do the same old chestnuts, sometimes reworded a bit. The boat problem becomes the car problem and this is succeeded by the airplane. Already it is time for a new one like this: "If one jet plane starts from Los Angeles for New York at 1900 miles per hour and another starts at the same time from New York for Los Angeles at 1950 miles per hour, where will they pass each other?" As soon as this is in print it will be time for rockets to the moon and back.

Now it is admitted that such problems serve a purpose. The lover of mathematics for its own sake, the student who is intending to specialize in the subject, enjoys working such puzzles and profits by formulating them and solving them. But to require them of a large number of pupils is about as rational as it would be to require all to take work leading to law or to medicine, because we need lawyers and doctors as well as mathematicians.

General education is concerned with teaching the things which should be common knowledge among an intelligent citizenry. General education should come before specialized education. It should prepare the youth for successful living, so that the very large number who cannot go on to liberal higher education possess a solid background of useful information as they move into the world of affairs. At the same time, general education lays the foundation for further specialized study, for the fortunate ones who can go beyond the junior college. General education should begin in the public schools and continue at least through two years of college.

I am concerned here with some ideas relating to a three semester-hour course to be taken by all freshmen, except those who are excused by "testing out." C. V.

[1] *Childhood Education*, XIX (January 1944), 220.

Newsom[2] points to some of the "growing pains" incident to attempts at developing such a course. For example, a course built around a unifying idea—that of function or that of a postulational study of the foundations of arithmetic—was unsuccessful. On the basis of years of experience, I would expect it to be.

The difficulties faced in trying to construct a course in general mathematics are great. There is fair agreement among mathematicians as to the proper courses for an engineer. He will need plane and solid geometry, algebra, trigonometry, analytic geometry, and calculus. But this is for the specialist. What should we offer those who plan an education not requiring specialized mathematics? Here there is no consensus, but many of us feel dissatisfied with our attempts thus far. After all, the idea is relatively new. We used to think that we should give all students the same beginning courses, and comforted ourselves with the thought that if they never encountered a quadratic or needed to rationalize a denominator, still the mental training was good for them. We felt, too, that there was cultural value in the acquiring of tools and language of mathematics even though little use of the algebra would be made by most students in after life. We mathematicians still believe in the cultural value of all mathematics. This value would be much greater if all teachers could present it as it is said President Harper of Chicago used to teach Hebrew—like a series of hairbreadth escapes. But many of us are sure that a much better selection of topics in mathematics can be made for the majority of students. I believe that not enough attention has yet been given to selection of topics and problems.

The young American is living in a world quite different from that of sixty years ago. He is used to the applications of electricity in many forms—electric lights, refrigerators, washing machines, sound tracks. He knows about Diesel engines, guided missiles, and jet planes. He hears about taxes, charge accounts, world trade, and national debts. While the world has completely changed, he cannot but be uneasy about, or impatient with, or indifferent to, the kind of problem which has no relation to reality.

Historically, a great deal of mathematics was invented in answer to definite needs. But in general, the needs were apparent to a few specialists or geniuses, scholars whose business it was. We can say that mathematics was not distributed among the people. Today it is desirable that a selected portion of mathematics should be. On the one hand, this portion need not be a collection of museum pieces. On the other hand, little help comes from a poll asking dentists, storekeepers, lawyers, and housewives, what mathematics they find useful. Nor do we achieve our goal by consulting a committee made up only of teachers of mathematics. Possibly such a committee would come closer to the mark if each member had time and interest enough to become familiar with other areas of general education—natural and biological sciences, communications, humanities, social sciences, logic, vocational orientation, health education, creative arts. In planning a mathematics program for all students on the junior college level, the entire faculty, as well as the mathematics staff, should have an interest. We are engaged in educating the total personality rather than proving the importance of mathematics. While we are setting up our standards and displaying our wares we must take care lest the parade of progress pass us by and leave us in an empty street. For the purposes of general education, our colleges are still too much departmentalized. The contributions of mathematics as a way of thinking, as an instrument in communications, as a method of bringing order out of chaos, as a necessity in conducting personal and community affairs, permeate all human living.

[2] "A Course in Mathematics for a Program of General Education," THE MATHEMATICS TEACHER, XLII (January 1949), 19.

So it seems to me some of the methods and values in mathematics should permeate general education in a planned manner. In communication we can show how the brevity and beauty of mathematical expression can aid in thinking and in communicating thought. A great scientist once wrote, in effect, that if you can't say it in numbers, you can't say it at all.

In logic and critical thinking—clear thinking—the mathematical approach aids in sound argumentation, in analysis of news, in the avoidance of hasty conclusions. In science the tie-in is so apparent that a few schools attempt an integrated course in which the mathematics is incorporated into the material. Biometrics is a well recognized science. In the humanities it can be pointed out that many philosophers were also acquainted with mathematics and used it freely. Art and music, sculpture and architecture, rely upon mathematical principles.[3] Economics has developed a mathematical approach, so that econometrics, handling economic problems by analysis through numbers, is important.

Space forbids mention in detail of the place of mathematics in citizenship and the social studies, in home and family living, in health, physical education, and recreation, in psychology and personal adjustment. Indeed, the contributions of any area in general education to other areas have not been sufficiently recognized, and the integrating process has hardly gone beyond an integration among courses in a given department.

In mathematics itself, we who consider ourselves subject matter specialists have done a lot of this kind of integrating. On the college level we have mixed up arithmetic, algebra, trigonometry, and analytic geometry. An example of this kind of integration is the three-volume *Fundamental Mathematics*,[4] published by the mathematics staff of the University of Chicago. This text is directed at liberal education and in our state-supported institutions would be a difficult course for seniors. Moreover, I would designate it as specialized mathematics, not adapted to mathematics in general education.

We have produced socialized mathematics for the public school. We have invented attractive names like "Living Mathematics." But we are still specialists and we have been attacked by people who wrote articles such as "Farewell to Algebra."[5] We have worried because in some states mathematics appears to be going the way of Latin—namely, out.

In this connection we have learned in general education that in certain teaching areas, such as music in the humanities, the nonspecialist achieves better results than the specialist. It would sound like heresy for a mathematician to assert that such would be the case in his field, and indeed I have no evidence that it would. Rather, I am saying that to prepare a course in general, or fundamental, mathematics, the mathematician should associate himself with teachers in other basic areas, visit their classes, examine their syllabi, look at their assigned reading, and at their tests.

I mention a few ideas, some of which have been suggested by the sort of approach mentioned. I have not attempted a logical order.

Mathematicians should examine extensive literature in addition to the syllabi, etc., mentioned. Very widely read are comic strips, sports news, and women's news. On a higher level above the daily newspapers, we find valuable material in the *Wall Street Journal*, in news and literary magazines. House organs and corporation reports give clues to a number of ideas in daily living. A list of materials is

[3] See, for example, George David Birkhoff, *Æsthetic Measure* (Cambridge: Harvard University Press, 1933).

See also T. M. Simpson, "Farewell Algebra," *The Journal of the Florida Education Association*, VIII (June 1931), 12–13.

[4] E. P. Northrup and others, (3rd ed. Chicago: University of Chicago Press, 1948–49).

[5] Simpson, *op. cit.*, p. 11.

easy to make. A universal American need is money and finance. The ramifications of this subject go far beyond the needs of fifty years ago—taxation, social security, public finance, installment buying. Another universal problem is home and family living. Books and articles on this subject are very numerous.

Objection will be taken to the approach in this paper on the ground that the mathematics staffs do not have time to acquaint themselves with other areas in general education. There is a measure of truth in this, yet until we find some way to overcome the difficulty we will continue to have a course which does not completely satisfy us, our students, or our colleagues.

I suggest an outline for a course in general mathematics for freshmen in a state-supported school. Arithmetic is universally needed by the average citizen, so I would begin with numbers. Difference of opinion immediately arises about the question of how far to discuss the growth and development. Other systems of notation besides the Arabic and the Roman could be left for assigned reading. In general, I would avoid lengthy computations and call attention to modern machines, IBM and other electronic calculators, and the role of the base 2. Much emphasis should be placed upon estimates and approximations. For example, if a student has $x/15 = 4/5$, he should know at once that x is less than 15. I am inclined to believe that inequalities are more important than equations or equalities, if for no other reason than that they are more numerous in life situations. Generalized arithmetic, or elementary algebra, can be introduced early and used sparingly. For example, in arithmetic, interest which is important is often treated by a separation into different areas. Very simple algebra enables the student to treat it in one formula, $I = prt$. Similarly for percentage and other topics. Proportion, or the rule of three, can be handled as an equality of two fractions. So in a college review and extension of arithmetic, it is possible to improve upon the usual public school course.

Throughout I would endeavor as far as possible to use only practical or sensible problems. As much emphasis should be placed upon *reading* and *interpreting* graphs as upon making them.

What to include in a brief treatment of geometry requires careful thought and good judgment. The benefits for critical thinking of a formal course in plane geometry are well known. If the student has not had geometry in high school, we must do the best we can with him.

Care, too, is necessary in dealing with measurements and in an introduction to a few fundamental ideas in statistics.

The Fundamental Mathematics course at Henderson State Teachers College, Arkadelphia, Arkansas, is divided into eight units:

1. The concept of number, its growth and development.
2. Our system of notation.
3. The symbols of mathematics and their meanings.
4. Inexact mathematics.
5. Percentage and its uses.
6. Installment buying and planned savings.
7. Making and using graphs.
8. The application of geometry to measurement.

With each unit there is a list of required readings and a list of recommended readings. The required readings are broken down into lessons. Each lesson lists a set of concepts to be acquired and a set of problems to be solved.

After teaching the course and observing it taught, I still think that this is a good outline, but it is probable that we could cover the dry bones more effectively.

The purpose of this course should not merely be to help fit the student to his environment. As Hutchins[6] aptly points out: "The notion of educating a man to

[6] Robert Maynard Hutchins, *The Higher Learning in America* (New Haven: Yale University Press, 1936), p. 66.

live in any particular time or place, to adjust him to any particular environment, is therefore foreign to a true conception of education." If we were able to foresee future changes, we might direct educational efforts intelligently. The material and ideas of the present must be used, but the mathematics we teach will deal with basic ideas and processes sure to be of value in the environment of the future. Since man is a time-binding animal and a thinking one, the relevance of mathematical ideas is eternal.

Malcolm McLean,[7] first Dean of the General College at Minnesota, points out that the field of mathematics is wide open for investigation as to the concepts, materials, and methods needed for general education, and that we work within too narrow limits. In the treatment of time, space, and mass, we have boasted, perhaps too confidently and smugly, that mathematics is one subject at least which is purely objective and free of connection with personality. Yet the concepts of mathematics, aside from its use as a tool, are fundamental. Few people are as adept, for example, at personal timing as the program directors for radio and television. The concept of an ordered set of subjects might be helpful to a writer or speaker. The writer of this paper is presently engaged in a time-mass-space personal experiment on a small scale (in fact, the bathroom scale), in which by calorie counting and controlled eating he is reducing to lower terms. This sounds facetious, but nevertheless more damage is done to the American people by too many calories than this world dreams of. Personal time measure is something to learn. Social timing—when to act in relation to others, the orderly planning of actions and of speaking—can profit from mathematical concepts.

To inspire and uplift our souls we have the rhythm, the music, the poetry of mathematics, its contribution to esthetic measure. If the essence of general education lies in the basic needs and wants of man, then mathematics has a much larger potential contribution than figuring taxes, keeping a budget, understanding the national debt, or appreciating the measure of light waves, as important as these are.

The lack of harmony, orderliness, and vision in the life of the average man desperately calls for the spirit as well as the body of mathematics. What profits it for a pupil to know how to find the number of arrangements of n things r at a time if he cannot also arrange the affairs of his living and growing?

All this is to say, perhaps, that every teacher of mathematics should be constantly reaching out for deeper, richer meaning in his subject. If it be objected that when the skills and techniques are taught, there is no time for looking further, the answer is that there have been and are inspired teachers who can show us the way. We can experiment, research, and explore fresh approaches as they have done. If Pythagoras could see in numbers the clue to music, if as Jeans remarks, "God continually geometrizes," then maybe we can have all this and Heaven too.

[7] H. T. Morse, ed., *General Education in Transition* (Minneapolis: The University of Minnesota Press, 1951), p. 42.

Student achievement of functional competence three years after completing algebra or general mathematics[1]

JOHN W. RENNER, *Director of Radiological Defense School, Federal Civil Defense Administration, Battle Creek, Michigan.*

Data are analyzed to see if some light can be shed on whether algebra or general mathematics best attains functional competence objectives.

THE WRITER recently completed a study of the mathematics course patterns which 1227 high school seniors (from forty Iowa public high schools) had studied during the first seven semesters of their high school careers and the level of measured functional competence in mathematics which they achieved on a standardized examination. A total of 237 (19.32%) of the 1227 students had no training in mathematics beyond one year of algebra or general mathematics as ninth graders. Consequently, it seems proper to ask this question: Is there a significant difference between the level of measured functional competence in mathematics achieved by twelfth graders who studied only algebra (as ninth graders) and those students who studied only general mathematics (as ninth graders) when scholastic aptitude is statistically controlled?

The arithmetic means (in standard score units) of the scores which the students in each group received on the *Davis Test of Functional Competence in Mathematics*[2] were interpreted as being the "levels" of functional competence[3] which these students had attained. This mean was 116.56 for the algebra group and 107.94 for the general mathematics group.

The arithmetic means (in standard score units) of the composite scores which the students in each group received on the *Iowa Tests of Educational Development*[4] were used as measures of scholastic aptitude. The composite score on the Iowa tests was used as a measure of scholastic aptitude because it was available for all the participants, and a study[5] done by the Examination Service of the State University of Iowa has shown that there are significant correlations between measures of scholastic aptitude and the composite score on the Iowa tests.

The mean composite scores on the Iowa tests (in standard score units) were 16.92 and 13.23, respectively, for the algebra

[1] This article is based on research done by the author at the State University of Iowa under the direction of Professor H. Vernon Price and Professor L. A. Van Dyke.

[2] World Book Company, Yonkers-on-Hudson, New York, 1951.

[3] The definition of functional competence in mathematics was given by the Commission on Post-War Plans ("The Second Report of the Commission on Post-War Plans," THE MATHEMATICS TEACHER, XXXVIII, [May 1945], 195–221). Since the Davis test is based upon this definition, it was selected as the instrument to measure the level of functional competence attained by the students in the study.

[4] Science Research Associates, Inc., 57 W. Grand Ave., Chicago 10, Illinois, 1953.

[5] *Relationships Among the Tests Given to Students Who Enter the College of Liberal Arts* (State University of Iowa, University Examination Service, Technical Bulletin No. 4 [Iowa City, April 1949]), p. 2.

and general mathematics groups. When scholastic aptitude was statistically controlled by the method of the analysis of covariance, the Davis test means (in standard score units) were 114.56 and 110.47, respectively, for the algebra and general mathematics groups. When the hypothesis of no difference between these means was tested (using a *t*-test), it was found that the difference (4.09) was statistically significant at the one per cent level of confidence in favor of the algebra group.

Thirty-one (77.50%) of the forty schools which participated in the research taught general mathematics. Eighteen schools (45.00%) used general mathematics exclusively as a remedial course in the ninth grade; i.e., the academically retarded students in mathematics were enrolled or encouraged to enroll in general mathematics. This practice was found to be more common in the larger schools than in the smaller schools. In other words, of those schools (77.50%) which consider general mathematics important enough to teach, 58.06 per cent of them consider its function to be that of a remedial-type course. Only five (12.50%) of the forty participating schools adhered to the type of general mathematics based on the definition of functional competence given by the Commission on Post-War Plans and recommended by the *Iowa State Course of Study for Mathematics*.[6] The remaining eight schools in which general mathematics was taught did not use it exclusively as a remedial course, nor did the type of general mathematics taught by those schools adhere to the recommendations of the course of study. In these schools the general mathematics course was based upon a single textbook.

On the basis of the foregoing data, it was concluded that for students who study only two semesters of mathematics, general mathematics when designed as a remedial course does not produce as great a degree of functional competence (as measured by the Davis test) as does algebra. It would have been desirable if a comparison could have been made between the achievement of the students who had studied only the type of general mathematics recommended by the Iowa course of study and the achievement of those students who had studied only algebra. Unfortunately, the data were not available to make this comparison, because of those students who had studied only two semesters of mathematics only nine had completed but two semesters of the type of general mathematics recommended by the Iowa course of study. The writer believes that there is need for a study which would make this comparison.

[6] Iowa Secondary School Cooperative Curriculum Program, Volume XIX, *Mathematics Series*, (Des Moines: State of Iowa, 1949).

A DATE TO REMEMBER

Thirty-Fifth Annual Convention

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• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Recent discoveries in Babylonian mathematics 1: zero, pi, and polygons

One of the best summaries of Babylonian mathematical achievements is to be found in the late Raymond Clare Archibald's article on this topic in THE MATHEMATICS TEACHER for May 1936. In this note he paid particular honor to Otto Neugebauer, the young Austrian scholar who, possessing the rare combination of a fine knowledge of mathematics, mechanics, and astronomy with the ability to read Babylonian cuneiform symbols, had been responsible for many of the discoveries listed there. Archibald's concluding statement was,

> Eight years of work by a young genius standing on the shoulders of great pioneering scholars [such as Frenchman F. Thureau-Dangin and H. V. Hilprecht of the University of Pennsylvania] have in extraordinary fashion greatly advanced the frontiers of our knowledge of Babylonian mathematics. One can not help feeling that the inspiration of such achievement will cause more than one man to shout, "Let knowledge grow from more to more," as *he* too joins in the endless torch race to
>
> "pass on the deathless brand
> From man to man."[1]

Since 1936 a war has intervened, but Otto Neugebauer has continued to unveil new aspects of Babylonian mathematics and astronomy even while moving from Germany to Denmark to the United States and here taking the time to serve as the first editor of *Mathematical Reviews*. The United States was fortunate indeed to have available a man who had previously founded and edited two German abstracting journals at a time when it became apparent that although we were not yet in the war, we were being cut off from those journals which were so important to a critically needed acceleration in the development of American mathematics, especially applied mathematics. We in America owe much to Otto Neugebauer, who is now chairman of the Department of the History of Mathematics at Brown University, for his services in setting up and for many years editing the now well-established and flourishing *Reviews*. However, all mathematical historians owe him a debt for his continued research and the postwar publication of two new books on Babylonian mathematics.[2]

The first of these contains several remarkable new discoveries, among which are the use by the Babylonians of an averaging or division method for approximating square roots and a table of Pythagorean triples which have been discussed in earlier numbers of this journal.[3]

The purpose of this note and its successors is to report several additional recent discoveries due to Neugebauer, and his collaborator A. Sachs, the Netherlander; M. E. M. Bruins, who has been studying

[1] "Babylonian Mathematics with Special Reference to Recent Discoveries," THE MATHEMATICS TEACHER, XXIX (May 1936), 219.

[2] O. Neugebauer and A. Sachs, *Mathematical Cuneiform Texts* (New Haven: American Oriental Society, 1945); O. Neugebauer, *The Exact Sciences in Antiquity* (Princeton: Princeton University Press, 1952).

[3] P. S. Jones, "$\sqrt{2}$ in Babylonia and America," THE MATHEMATICS TEACHER, XLII (October 1949), 307, and "The Pythagorean Theorem," THE MATHEMATICS TEACHER, XLII (April 1950), 162.

materials located at Susa, the capital of Ancient Elam, by a French archaeological expedition in 1936; and the Arab Taha Baqir, who has been studying tablets recently found at Tell Harmal in Iraq. In this note we shall turn our attention to three topics only: *zero*, π, and *polygons*. For persons who would like a broader survey of Babylonian writings and Babylonian and Egyptian mathematics, we recommend in addition to Archibald's article and the more technical references cited here, two recent "pocket books," V. Gordon Childe's *Man Makes Himself*[4] and Edward Chiera's *They Wrote on Clay*.[5]

It is rather remarkable that although the importance of zero is recognized the world around, its history is still clouded with doubts and uncertainties, and the concept of zero and the operations with it remain pedagogical stumbling blocks through the grades. Van der Waerden writes, "The zero is the most important digit. It is a stroke of genius, to make something out of nothing by giving it a name and inventing a symbol for it." He quotes Halsted as saying "It is like coining the Nirvana into dynamos," and elaborates Freudenthal's theory that the Hindus developed if not their whole decimal system, at least their zero concept and a round symbol for it out of the Greeks' round symbol for zero. This concept the Greeks, in turn, had obtained from the Babylonians, he believes.[6]

Neugebauer agrees with this when he writes, "It seems to me rather plausible to explain the decimal place value notation [of the Hindus] as a modification of the sexagesimal place value notation with which the Hindus became familiar through Hellenistic Astronomy."[7] He cites further evidence for this line of communication from the history of astronomy.

The facts about zero at the moment seem to be these: There was no zero in use in old Babylonian times (1800–1600 B.C.). One had to determine the sexagesimal places where we would have written a zero either by the context of the problem or, occasionally, by the spacing of the symbols. In the later Babylonian or Seleucid period (300 B.C.—0), a special symbol, which was also used as a separation mark between sentences, came into use for a zero in both mathematical and astronomical texts. Further, there is a definite possibility that the Babylonians used this mark for a zero within a number, as early as the end of the eighth century B.C.[8]

The Greeks took over Babylonian data and sexagesimal fractions when Hipparchus began to study and write about astronomy. In fact, we today get our minutes and seconds via the Arabs from this Greek use of Babylonian sexagesimal fractions, into the writing of which the Greeks introduced their own symbols for zero.[9] One of these was a round symbol which van der Waerden and others have regarded as coming from the first letter of the Greek work for *nothing*. Neugebauer doubts this and cites many other symbols used by early Greeks for representing a zero within a number, but agrees that the later Greek astronomer Ptolemy (c. 150 A.D.) was the first to write a zero at the *end* of a number. For this he used a circular symbol.[10]

There seems to be no evidence that the Babylonians ever regarded zero as a num-

[4] (New York: New American Library of World Literature, 1951.) This paper-backed Mentor Book was originally published in England in 1936 and revised in 1941 and 1951. Chapter VIII, "The Revolution of Human Knowledge," deals fairly extensively with Egyptian and Babylonian mathematics.

[5] George G. Cameron (ed.), *They Wrote on Clay* (Chicago: University of Chicago Press, 1938); reprinted in 1956 as a paper-backed Phoenix Book. Little space is devoted to Babylonian mathematics (pp. 154–157), but among its profuse illustrations are good pictures of tablets containing maps (p. 160) and "blueprints" (p. 163).

[6] B. L. van der Waerden, *Science Awakening* (Groningen, Holland: P. Noordhoff Ltd., 1954), p. 56.

[7] *The Exact Sciences*, p. 180.

[8] Neugebauer, *The Exact Sciences*, pp. 16, 20, 26, and *Mathematical Cuneiform Texts*, p. 34.

[9] P. S. Jones, "Angular Measure—Enough of Its History to Improve Its Teaching," THE MATHEMATICS TEACHER, XLVI (October 1953), 419.

[10] *The Exact Sciences*, pp. 13–14.

ber by itself which could enter into operations with other numbers. Boyer presents good evidence that this distinction belongs to Aristotle, who discussed division by zero in connection with speed through a vacuum as compared to speed through a resisting medium. Brahmagupta (Hindu, c. 600 A.D.), who has often been given credit for the first discussion of division by zero, did write of it more explicitly and was probably the first to discuss the division of zero by zero.[11]

On page 59 of his *Mathematical Cuneiform Texts*, Neugebauer, after offering a rather involved and tentative explanation of the multiplier used on a tablet dealing with the volume and circumference of a log, remarks that the result would have been obtained directly if one had assumed $\pi = 3;7,30$. This is his notation for the sexagesimal number

$$3+\frac{7}{60}+\frac{30}{60^2},$$

which would have been, decimally, 3.125 or $3\frac{1}{8}$. He then continues "This assumption, however, seems unwarranted because this approximation of π is not attested elsewhere in Babylonian mathematics."

This is not the first time that Neugebauer has anticipated new discoveries. In fact, he had speculated that the Babylonians had probably made advances in number theory some time before the discovery of the Plimpton tablet containing a table of Pythagorean triples. In the case of π, Neugebauer was able to add a note after the completion of the manuscript of *The Exact Sciences in Antiquity* pointing to a tablet newly published by Bruins. Here he found a correction factor which the Babylonians inserted into their older processes when a more accurate determination of area was needed.

The Babylonians in general determined the circumference of a circle by multiplying its diameter by 3, this is equivalent to saying $C = \pi d$, with $\pi = 3$. They found the area as one twelfth of the circumference squared. In modern symbols, this amounts to saying $A = \frac{1}{12}C^2$ which, if π were 3, would have been equivalent to $A = C^2/(4 \cdot \pi)$, a correct formula if the correct value of π were used. Bruins reported in 1950 a tablet excavated by French archaeologists in 1936 on which it was directed that for more accurate results the $\frac{1}{12}$ should be multiplied by ;57,36. This means

$$\frac{57}{60}+\frac{36}{60^2}$$

which is $\frac{24}{25}$. Since the correct multiplier would be $1/4\pi$, and since the Babylonian corrected value would have been

$$\left(\frac{24}{25}\right)\left(\frac{1}{12}\right),$$

by equating these two values and solving for π, we find that this Babylonian process for determining the area of a circle was equivalent to using $\pi = 3\frac{1}{8}$!

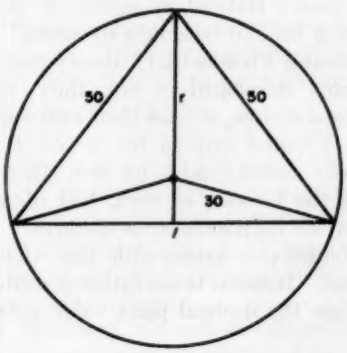

Figure 1

These tablets discussed by Bruins also contain a problem in which the radius of the circle circumscribed about an isosceles triangle with sides 50, 50, and 1, was determined.[12] These peculiar dimensions

[11] C. B. Boyer, "An Early Reference to Division by Zero," *The American Mathematical Monthly*, VI (October 1943), 487–91.

[12] M. E. M. Bruins, *Nouvelles Découvertes sur les Mathématiques Babyloniennes* (Paris: Université de Paris, 1952), pp. 18–21.

need to be interpreted in terms of the sexagesimal system in which one half of 1 is 30 i.e., if, lacking a terminal zero, 1 really represented 1, 0 in this case, i.e., one sixty and no units, then one half of it is 30. Or if 1 really means 1, then one half of it is $\frac{30}{60}$, which the Babylonian would write as ;30 (using the corresponding cuneiform symbols of course).

Figure 1 shows the diagram. The altitude is 40, and the Babylonians found r by the process which we would symbolize as solving the equation $(40-r)^2+30^2=r^2$.

This and some problems on regular polygons indicate that the Babylonians went farther into more theoretical geometry than has been previously realized. For example, a table lists good values for the coefficients which multiplied times the square of a side give the area of a regular pentagon, hexagon, and heptagon.

These coefficients are in agreement with the ones given by Heron (c. 75 A.D.) in his *Metrica* for determining the sides of the regular inscribed pentagon, hexagon, heptagon, or nonagon, by multiplying the radius of the circle by 6/5, 6/6, 7/8, and 2/3 respectively. Bruins suggests that the first two and last of these might have been derived by the Babylonians by processes equivalent to the formula $S_n=6/nr$. This could have been derived by assuming that the perimeter p_n of the $n$sided polygon was nearly equal to the circumference of a circle. Thus $p_n=S_n \cdot n \approx 2\pi r$, and then using the Babylonian approximation, 3, in place of π. Heron derived the ratio 7/8 for the heptagon by using for the side of the heptagon the altitude of the central triangle associated with a hexagon and by using 7/4 as an approximation to the $\sqrt{3}$. This still gives a good approximate construction for a heptagon which, of course, we now know to be nonconstructible with ruler and compasses.

These recent discoveries show, then, that the Babylonians had gone farther in geometry than has been previously known. However, it is still true that their most remarkable and extensive achievements were algebraic.

Enrollment data—1954

It should be recalled that the number of high school pupils has increased rapidly. The percentage of pupils in a subject may have remained constant or even decreased while their number actually increased. For example, the percentage of pupils enrolled in trigonometry has remained rather constant, but the number has increased fifteenfold since 1900. Perhaps a less distorted picture of mathematics enrollments is given by using as a base the enrollment in the grade where the course is normally offered rather than the total enrollment in the last four years of high school.—*Pamphlet No.* 118, *U.S. Department of Health, Education, and Welfare.*

Edited by Lucien B. Kinney, Stanford University, and Dan Dawson, Stanford University, Stanford, California

An area of neglect in the study of arithmetic —mental arithmetic[1]

by Gerald Brown, Los Angeles State College, Los Angeles, California

The mental arithmetic issue

The mental arithmetic theme is not a new one. Emphasis on the importance of mental arithmetic—that is, computation without the use of pencil and paper—can be found in the literature of any period.

David Eugene Smith said in his book, published in 1909, "The ordinary purchase of household supplies requires a practical ability in the mental arithmetic of daily life. . . . Therefore, it will not do . . . to drill children only in written arithmetic if we expect them to be reasonably ready in purely mental work."[2]

And ten years later, Guy M. Wilson reported in his study published as a Teacher's College Contribution, ". . . this study shows that adults do not figure through complete situations, such as budgeting the family income, determining by careful estimates in advance, the desirability of buying a carload of steers and finishing them for market. . . . "[3]

Wilson's implication here is one that is borne out daily in our own experience—that the arithmetic of daily life, the arithmetic of the mathematically literate citizen, is not primarily the arithmetic of pencil and paper computation, but a form of mental computation which is at times accurate, but probably more often approximate.

You mentally compute what time to leave your home to be on time for an appointment. You mentally compute how much money you will need, as you leave your home to attend a convention. You pace off your lawn and compute, mentally, approximately how much grass seed or weed killer you need to purchase. You look at your gas gauge and estimate whether the supply is adequate to propel you to the next town.

Yes, the extensive use of mental arithmetic cannot be challenged as the arithmetic for daily living. From the farmer, who mentally approximates the tonnage of hay in a stack, to the housewife, who mentally allocates the month's food budget, to the youngster, who decides between the popsicle now or the ball glove later, the activities of our citizens are influenced by approximate or accurate results from mental computations.

That mental arithmetic has extended application in practical affairs is not a matter of personal opinion. It has been

[1] A paper presented to the Junior High School Section at the annual conference of the National Council of Teachers of Mathematics, August 19, 1956, in Los Angeles.

[2] David Eugene Smith, *The Teaching of Arithmetic* (Boston: Ginn & Co., 1909), p. 55.

[3] *Survey of the Social and Business Usage of Arithmetic* (Teachers College Columbia University Contribution to Education No. 100 [New York, 1919]), p. 56.

verified at various times in our research, by Wilson as early as 1919, in the study referred to earlier, and as recently as this year at Los Angeles State College, where a study is underway to investigate the uses of mathematics by adults in their everyday activities. A sample, taken at random from the large number of cases in the study, shows that 77 per cent of the arithmetic computation performed by this group was performed mentally.

Does the teaching of arithmetic in our schools reflect the use that is to be made of it in later life? Spitzer raised this issue in 1948, when he said in his book. *The Teaching of Arithmetic.* "Fortunately, most of the computations of everyday life involve calculations that do not require the use of pencil and paper. In order to be as nearly lifelike as possible, therefore, the learning exercises in school arithmetic should make extensive use of oral or unwritten calculation."[4]

Current recognition of the problem by leaders in the field is indicated by Irene Sauble's article in *The Arithmetic Teacher* of April, 1955. She states, "At the present time in all arithmetic programs, increased emphasis is being placed upon development of ability to compute mentally and to obtain reasonably close approximate answers for computations in all four processes."[5]

This "increased emphasis" referred to by Sauble has not taken any substantial form in the classroom, however. Discussions with teachers and visits to schools reveal that, although some exercises related to mental computation are appearing in the textbooks, it is still receiving only minor emphasis. A careful study of achievement tests in arithmetic fails to divulge an earnest attempt to evaluate the ability to approximate results or compute mentally. We are forced to conclude that objectives not covered by evaluation of some sort remain relatively unimportant.

[4] Herbert Spitzer, *The Teaching of Arithmetic* (New York: Houghton Mifflin Company, 1948), p. 52.

[5] Irene Sauble, "Development of Ability to Estimate and Compute Mentally," *The Arithmetic Teacher*, II, (April 1955), 33.

How mental computation takes place

Having made a case for the importance of nonpaper-and-pencil ability in arithmetic, then how can this competence be developed in the classroom? To suggest or expect a decrease in emphasis on the formal work of arithmetic in view of current conditions would be folly, although no more so than to expect that ability to handle quantities mentally can be developed incidentally from such formal study. Perhaps an examination of how and under what conditions mental calculations are performed would be appropriate before devising plans to develop competence.

Some research into the processes involved in approximation or mental computation would be extremely helpful here. However, since there is not a definitive study in this area, let us frame an hypothesis that fits the evidence available.

A reasonable hypothesis is the following: Mental calculations of an exact or approximate nature are carried on through intelligent utilization of a known product or quotient and adjustment of the answer from there. These known products or quotients will be referred to here as *reference points.*

Thus the answer to 51×52 can be approximated by using 50×50 equals 2500 as a *reference point* and adjusting that answer upward. 3625 divided by 38 can be approximated by using 3600 divided by 40 as a convenient *reference point* with appropriate adjustment of the answer.

Frequently, problems arise which fit a *reference point* exactly, and thus an exact answer can be found. For example, $37\frac{1}{2} \times 32$ works out easily and exactly by using aliquot parts of 100 as *reference points.*

Obviously, the mathematical sophistication of the individual determines the extent to which this sort of mental calcula-

tion can be done. An analysis of the mental arithmetic process reveals three distinct areas where training is requisite:

1. A knowledge of reference points;
2. Skill in selection of the proper reference point;
3. Skill in adjusting the initial answer to an improved answer.

Illustration: A man wishing to buy lawn seed determined by measurement the dimensions of his yard to be 53′×48′. Being unable or unwilling to multiply these two numbers mentally, he searched for a convenient reference point. Since one dimension is slightly below and the other slightly above 50, he used 50×50 as an approximation, arriving at 2500 square feet for the area. Since 53 is 3 larger than 50, and 48 is only 2 less than 50, it was necessary for him to adjust the 2500 upward, and a logical figure was one more 50, which produced an answer of 2550 square feet—a figure sufficiently accurate for the purpose of purchasing seed.

Note that in the illustration, the gentleman selected an appropriate reference point and made an adjustment to his initial answer.

Developing competence in mental arithmetic

The implication here for the teacher seems quite clear. Prospective citizens need to extend their knowledge of reference points and to have opportunities to develop skill in their use.

Let us examine several multiplication processes which can be used as reference points. The multiplication tables themselves are reference points, and all approximation involving multiplication and division stems from them. For example, since we know that 8×9 equals 72, we can mentally multiply 8×18—since 18 is twice 9, our answer is twice 72, or 144.

The most convenient reference points are probably multiplying or dividing by quantities ending in zero. We multiply 50×60, obtaining 3000, having learned that we need only multiply the 5×6 and annex two zeros. It should not be forgotten, however, that these seemingly obvious facts need to be learned. There is probably nothing more frustrating to a teacher than to have a student divide by 100, utilizing the long division algorism meticulously going through each step.

What other reference points are there? We have all developed, through experience or reason, such concepts as multiplication by 25, whereby we divide by 4 and annex two zeros. Similarly, multiplication by 50 can be effected by dividing by 2 and annexing two zeros.

The mental method of squaring a number ending in 5 can be interesting, recreational, and useful as an extension of our reference points. Convenient systems exist for squaring any number between 25 and 75, for dividing or multiplying by 125, for adding unit fractions, or multiplying two-digit numbers when the units digits add to ten, and the tens digits are identical. However, it is not our purpose here to examine in detail the various types of problems that can be worked mentally. We all know several types and all have our favorites—it is enough to point out that they exist in quantity and can be advantageously used as reference points to extend our pupils' abilities to compute mentally. For further discussion of uses of mental computation in addition, the reader is referred to Sauble's excellent article mentioned previously.

How would a teacher operate in the classroom in attempting to develop competence in the area under discussion? In view of the points made thus far, the following plan seems feasible:

1. Most of the reference points referred to are extremely interesting to the uninitiated, who look upon them as short cuts to an arduous process. Thus they can be introduced to a class as a recreation and used in informal situations as extensively as possible without defeating the natural pupil interest.

2. As in all concept development, the technique needs to be examined to see why

it acts as it does. What properties of the number system are involved? What other operations are related to it? And for junior high school grades, one would ask "Can it be generalized further?"

3. At this point, in accordance with current learning theory, as set forth by Kinney and Purdy's "flow chart,"[6] the good teacher would use the concept in a structured situation. Here children would identify exercises where the concept applied and would mentally supply the answers.

4. The teacher has now set the groundwork prerequisite to use of the short cut as a reference point. To continue, she will need a set of word problems drawn from children's experiences which involve the process in the concept being developed, however using numbers slightly removed from ones which fit the process.

Let us illustrate these steps concretely.

Step 1:

66	33	81	52	47	35	19
×64	×37	×89	×58	×43	×35	×11

The teacher gets the attention of the class by saying, "Here are some exercises I can do mentally," and proceeds glibly to reel off answers. After sufficient demonstration to arouse interest, the teacher identifies problems in which the short cut works and explains how it is done. This particular short cut requires the tens digits of multiplicand and multiplier be identical, while the ones digits add to ten. If $n=$ the tens digit, the correct product is $100n(n+1)$ plus the product of the units digits. Example: $66\times64=100(6\times7)+(6\times4)=4224$.

Step 2: One would hope that the class will ask why this process works. But if they do not raise the question, the teacher must. Mathematical operations are not magical nor mystical, but operate according to a plan and can be explained. This one, which probably would not be used below the fifth grade, could be explained meaningfully to an arithmetic class as arising out of our decimal system, and algebraically and just as meaningfully to an algebra or geometry class.

Step 3: The instructor now sets up a situation where many examples can be utilized in a drill situation. Some children make them up while others solve them. The more rapid students try something similar in three figures, while slower ones work with easier two-digit multiples.

Step 4: Finally, the teacher is ready to utilize the concept as a reference point, using problems like the following: "We have 28 children in our room. If each brought a 23¢ contribution for our party, would we have $6.20 for food?"

Here she points out that $28\times22¢$ can be used as an approximation, yielding $6.16. Since the adjustment would be upward, the 23¢ contribution is adequate.

Unquestionably, a program as just described will take time. However, this time will be extremely well spent. The activities proposed here can reasonably be expected to contribute to an understanding of mathematics, to supplement the regular drill program, and to help build desirable attitudes toward arithmetic and its application.

An assumption implicit in any such proposal is that each arithmetic classroom is staffed by an adequately trained teacher who is familiar with the number system and aware of many recreations—a teacher familiar enough with the processes, applications, and mental computations that teaching the steps previously described is no hardship. In view of current conditions, these abilities cannot be assumed. At the same time, realization of this deficiency re-emphasizes the need for every member of the profession to keep working toward recruitment of high caliber people and development of an adequate preparation program for arithmetic teachers at all levels.

[6] Lucien B. Kinney and C. Richard Purdy, *Teaching Mathematics in the Secondary School* (New York: Rinehart, 1952), p. 54.

Junior high school and the stock market

by Harriet Schultz, Pomona, California

In answer to the complaint of many parents today that their children are not learning arithmetic that will be of practical use, here is an arithmetic unit that the junior high teacher can use to enliven the classroom and at the same time develop many meaningful implications.

It is a common belief among today's citizenry that the buying and selling of stocks and bonds is a transaction veiled in mystery and indulged in only by big business tycoons. They believe the average middle class working people have no place in its scheme. However upon closer and more careful examination it is found that the great wealth upon which America is built is the result of the savings of millions of everyday Americans. Their savings have been put, in great part, into the stocks and bonds of national and local business and industry. What better place than these junior high years, when the boys and girls are just beginning to become aware of earning and saving money, to introduce how their savings may earn money for them? To the girls in the class who may believe that investing money will be of no future use or interest to them, merely point out that the bulk of America's investments today are held by women.

A bang-up way to initiate the unit is to tape record the stock market radio report and play it for the class. In the discussion that follows, find out how much pupils understood, then use this as a jumping off place for explanations. (This same diagnostic recording can be utilized at the end of the unit as a measure of the class's progress and give the children a chance to realize how much they have learned.) A little historical background and discussion of vocabulary will lead to the startling announcement that each class member will be given an imaginary $1000 for one month to invest in the stock market in whatever way he chooses.

After continued discussion on terms needed to know and lessons on how to read the daily stock market report in the newspaper, the class members select their investments. From then on, each day, the pupils bring in the market reports and the class notes their own stocks' current values. The trend of the stocks can be illustrated very well by a graph, kept weekly or monthly. Other activities that can be included are finding the percentages of gain and loss, oral reports by the pupils, and computing compound interest. A good motivation, when the class has had the necessary background work, is to have a local broker come in and talk to the group about their stocks. At the end of the month those who made the biggest and least gains might be evaluated by the class in an attempt to find reasons. A display of the graphs made in the unit is also a good culminating activity.

If well planned, organized, and taught by the teacher this unit will have been well worth its time. The class will have had practical applications of percentage, compound interest, fractions, and of course graph building. In addition, these junior high age children will have had a beginning acquaintance with learning how savings may earn money and how to evaluate safety of investments—necessary and valuable ideas to them as America's future investors!

EDITOR'S NOTE: In many communities it might be possible for children to check with their parents about stocks owned.

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

Not real, yet very useful. . .

Every once in a while we stumble upon a beautiful example of perfectly familiar ideas expressed in an unusually attractive manner. The most recent instance is the opening section of an engaging article[1] by Professor Billy E. Goetz of the Massachusetts Institute of Technology, entitled "The Usefulness of the Impossible," from which we quote through the courtesy of both the author and the editors. This material is quoted at some length not only because the ideas are significant—familiar enough, to be sure, even to the neophyte—but also for the charming simplicity with which they are set forth. Is it too much to hope that geometry teachers might handle such ideas with the same frank and lucid directness?

A straight line has no width, no depth, no wiggles, and no ends.

There are no straight lines. We have ideas about these nonexistent impossibilities: we even draw pictures of them. But they do not exist.

Ask a draftsman to draw a straight line. Place his product under a microscope. Observe the variable width; the darkness and lightness which mark its varying depth; the wiggles, both lateral and vertical, where the line plunges and climbs and wriggles among the fibers. Put the microscope away and observe the ends of the line where it runs off the paper. Finally, contemplate the curvature of space.

A straight line hasn't even a definition. Heath, in his *Thirteen Books of Euclid's Elements*, discusses the problem. Before Euclid, Parmenides has stated "straight is whatever has its middle in front of both ends." Euclid defined a straight line as "a line which lies evenly with the points on itself." Heron, in the first century, A.D., suggested "a line stretched to the utmost." Equally old, although restated by Leibniz and put into the following form by Gauss, is: "The line in which lie all points that, during the revolution of a part of space about two fixed points, maintain their position unchanged. . . ." While this may seem definitions aplenty, the modern view as expressed by Pfleiderer is: "It seems as though the notion of a straight line, owing to its simplicity, cannot be explained by any regular definition which does not introduce words already containing in themselves, by implication, the notion to be defined, as though it were impossible, if a person does not already know what the term straight here means, to teach it to him unless by putting before him in some way a picture or a drawing of it."

A point has no dimensions, no existence, and no definition. A picture of a point is a ragged area of uncertain extent on a rough surface. By refining the picture, the area diminishes. Finally, we can refine no more. We place the portraits side by side in the sequence of successive refinement. Then we point far to the side and say, "The picture that belongs there, where refinement has been carried to the ultimate and the dimensions have entirely vanished; that picture, if it existed, would be a true dimensionless point."

Euclid lists twenty-three definitions which define more than twenty-three figments of the imagination. Next he postulates an ability to draw straight lines from point to point, to project these straight lines indefinitely in either direction, and to draw circles; all manifest impossibilities. He assumes all right angles are equal, although there are no right angles. And he includes the famous postulate of parallels, by denying which Riemann created the geometry of curved space. Lastly, Euclid introduces five "common notions" as axioms; that is, as self-evident truths; the very first of which is impossible, let alone true. "Things equal to the same thing are equal to each other." No two real things are precisely equal. The common equal is doubly doubtful. It is an impossible ideal which can be approached only imaginatively.

He who protests that this is a quibble, that for practical purposes equals do exist, merely impales himself on the other horn of a dilemma. In a chapter on Number in his *Aspects of Science*, Tobias Dantzig tells of two bars, A and B, so nearly the same length as to defeat all attempts to ascertain which is the longer. Practically,

[1] *Bulletin, American Association of University Professors*, XLII (1956), 275–77. By permission.

they are of equal length. Another bar, C, is so nearly the same length as B as to defy all attempts to show a difference. Practically, B and C are also of equal length. But when A is compared with C, there is no difficulty in proving that C is longer than A. So if we deal with practical equalities, things may be equal to the same thing without being equal to each other.

The whole of geometry is consciously, wilfully, deliberately antagonistic to reality. In classical geometry, the compass and straight edge are allowed, the ruler forbidden. The compass and straight edge are both mystical, for they produce true circles and lines. The ruler is a practical tool used by artisans and beneath the dignity of a Greek philosopher-mathematician. Modern geometry has exceeded the purity of the ancients. It deals with points which are not points but vague unspecified items; lines which are not lines but classes of items; and planes that are not planes but classes of classes. In some modern geometries, straight lines are distorted geodesics twisting and wriggling in a warped and changing space. In modern physics, these writhing monsters are chopped into a large, but finite, number of tiny, but not infinitesimal, discontinuous, discrete quanta. This is as near as we can get to "real" straight lines!

Nor is geometry the black sheep of mathematics. All mathematics carries the family taint; all mathematics is a gigantic tussle with nonexistent impossibilities. We are cautioned not to add poems and railroad trains, or to subtract centimeters from miles. Without referring to what is added on the balance sheets of business enterprises, let us merely note that if one ought not to add apples and bananas, one probably shouldn't add Jonathans and Rome Beauties, or big apples and little ones, or 1929 dollars and 1932 dollars. If we are to add at all, we must add unlikes, in violation of all mathematical regulations.

I shall not attempt to prove that mathematics is useful. I will admit it and so save myself the trouble of proving that here is a great and respected discipline where all is impossible and yet much is useful. The usefulness largely flows from the impossibility. Mathematical concepts have been simplified and generalized until they describe an imaginative world no part of which could possibly exist outside men's minds. But their simplicity and generality have made them amenable to the laws of logic. We can think about them with sufficient rigor to build a truly impressive edifice, much of which translates into physics and engineering.

How straight is straight?. . .

. . . Gave his sentiment divine,
Against the being of a line;
Line in Nature is not found.
—Anonymous

No earthborn will
Could ever trace a faultless line;
Our truest steps are human still—
To walk unswerving were divine.
—Anonymous

No power on earth, however great,
Can pull a string, however fine,
Into a horizontal line
That shall be absolutely straight.
—Anonymous

These lines of verse came to mind upon reading an advertisement of the Eastman Kodak Company, in the July 1956 issue of *Science*, which told of a modern procedure used in connection with "awesome alignment problems." We pass along the pertinent part, with permission of the Eastman people, in the belief that it will be of interest to teachers of geometry.

No Knots

Take the Great Pyramid of Cheops at Gizeh. Take the Eiffel Tower. Take the *Nautilus*. Take one of those gigantic atom smashers. Take even a little thing like a million dollar turbine in a power generating station or a 70-foot planer bed. Always there comes a moment when the engineer-in-charge has to say, "OK, boys. She'll never be lined up any better than she is right now. Let's get on with the job." And the irrevocable next step is taken. Making a decision like that can take a lot out of a man.

Any gadget that puts such a decision on firmer ground ought to be worth quite a bit in peace of mind alone, to say nothing of the time cards of all those men standing around waiting, waiting for the word.

Right here we could make a big mistake by overplaying our hand. Let's better make plain where the new *Kodak Axicon* stands in relation to the art of aligning long axes.

The word "axicon" was coined by one of our chaps to designate a simple new class of axially symmetric optical elements, which, with the study of optics a couple of millennia old, he was lucky and smart enough to invent. An axicon images a point source of light along the axis as a straight line in space. No wire, however tight, can be so perfectly free of kink and sag. What of a telescope, you say?

A telescope objective forms its image at a different little knot in space for each successive target along the line of sight. In following these images with the cross hairs, there is a chance for error of parallelism between the focusing motion and the axis. There is also doubt about how much of the observed displacement is real and how much of it is parallax because of inability to locate the knots exactly. With an axicon there is no focusing. Anywhere along a length of 40 feet—100 feet or more, if you like—the line of light is equally thin, forms an equally hard little point of light where intercepted.

Upon inquiry, we gleaned further information about the axicon, briefly summarized below:

The obvious way of establishing a straight line, one still in use in many engineering applications, is to stretch a wire tightly between two end points. Measurement of distances from the wire then determines the position of intermediate stages. For example, if the centers of a series of bores are to be placed on a straight line, the wire is stretched to run through the centers of the two end bores, whereupon measurements from the wire to the inside surfaces of intermediate bores tell the relationship between these bores and the wire. This problem is common to the steam turbine and shipbuilding industries where successive bearing seats must be closely aligned. In aligning the bed on which the table of a large planer travels, the wire is likewise centered in the ends of the V rail, and measurements from the wire to the V indicate the departure of the V from the direction of the wire. In all cases where a wire is used as a straight line reference, care must be taken to avoid errors due to:

1. sag of the wire, however tightly it is stretched,
2. vibration of the wire,
3. small kinks in the wire which may not be readily evident, and
4. variation in the user's ability to determine the amount of interference in touching the wire with the measuring tool.

The difficulties in controlling these factors have caused manufacturing and maintenance people to use optical methods for alignment, and within recent years telescope aligners have been making significant contributions to the solution of alignment problems. The chief feature of the optical approach is the straightness of the line of sight; the direction along which an optical device sights is straighter than the wire since it does not sag or kink. Moreover, the determination of distance from the line of sight to surfaces can be made with greater precision.

The Kodak *azicon*, while preserving the accuracy of modern precision-built sighting telescopes, by its very nature avoids the potential errors of parallax and the focusing slide. The word "axicon" refers to a new type of optical element.

An axicon can take many different forms. One of the most useful of these is a piece of glass having one flat side and one conical side rather than the spherical curve usually found on a conventional lens. Like other types of axicons, this glass cone has some unique features. One of them is that it will image a small source of light into a whole line of images. Figure 1 shows this feature.

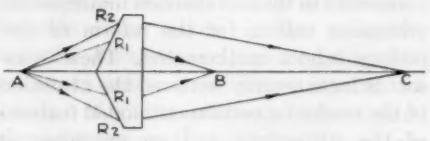

Figure 1

If (A) in Figure 1 is a point of light, such as an illuminated pinhole or a small tungsten filament, some of the light will strike the axicon at point R-1. The axicon will bend these rays so that an image will be formed at (B). Further, other rays from point (A) will strike the axicon at R-2, and the axicon will form another image at point (C). Thus, if a screen or a piece of paper is placed at point (B), an image of (A) will be seen, and if the screen is moved, an image of (A) will continue to be seen as the screen is moved toward (C). Actually, an image will be seen even if the screen is moved closer to the axicon than (B) or much farther away than (C). A "transmitting axicon," as this type is called, has an imaging range which extends from the axicon itself to infinity. Ordinary lenses, of course, cannot give this result.

Have you read?

WILSON, JACK D., "The Pre-Service Education of High School Mathematics Teachers," *California Journal of Secondary Education*, October 1956, pp. 330–36.

This article points out some of the deficiencies in the preparation of mathematics teachers. As teachers ourselves, we can do a great deal toward bringing about changes in preparation. You should read this article and see if you agree that there should be more advanced arithmetic, modern mathematics, logic, and statistics in the program. What should be done about field work, laboratory practice, and historical development? What should be the tactics and strategy of the instructor? How can he provide creative problems and dramatic demonstrations? What are our responsibilities for definitions and assumptions? The article gives an interesting discussion and leaves you with the feeling that the key of mathematics unlocks the door of progress.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

A column of unofficial comment

If it's good mathematics, it's easier to learn

by H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

Since World War II, we frequently find comments in the literature on mathematics education calling for the reform of secondary school mathematics. These comments have usually centered the attention of the reader on certain outmoded features of the curriculum and on practices, at times found even in college classes, which do not conform to good mathematical standards. These comments often present a rather severe criticism of the kind of mathematics taught in secondary school classes.

The present curriculum is being criticized by people who know mathematics. They are leaders in the mathematical world, and, hence, there is reason to give careful attention to what they are saying. In essence, the mathematicians are saying that mathematics has moved on from its position of fifty, or even twenty-five, years ago. There has been a change in the spirit of mathematics and a change in what is deemed to be important mathematics for today's social, economic, and technological problems. These suggested changes must be given serious consideration by those who are teaching secondary mathematics. It is good to know that the National Council has recognized that this is the time to reconsider the secondary and elementary mathematics program by appointing a Committee on the Secondary Curriculum[1] and a Committee on the Elementary Curriculum.[2]

There are mathematicians who contend that some of the mathematics taught in the high school is bad mathematics. As an example, they point to the mutiliated version of Euclidean geometry that is being offered in the secondary schools. To replace this kind of mathematics, we find that "modern" mathematics is being advocated. Others say that whatever mathematics is taught in the high school should be "honest" mathematics. No matter what adjective is used, the import is clear. The secondary schools must prepare for a change.

There is, however, at least one point about modernizing the curriculum which is frequently overlooked. *We need good mathematics in the schools because good mathematics is easier to learn than bad mathematics.* Since this point is a psychological one, and since those who criticize secondary mathematics most severely are mathematicians, it is not surprising to find that problems of learning are not given due weight. Even so, these problems exist and the answers to these problems can be a powerful force for stimulating a reform in mathematics.

There is only a minimum of research to support the thesis that "good mathematics is easier to learn," but common sense and experience do yield some support. It might be well to examine briefly the common-

[1] The members of the Secondary Committee are: Frank B. Allen, Chairman; Howard F. Fehr; John Mayor; Bruce Meserve; Sheldon Myers; Alfred Putnam; Marie Wilcox.

[2] The members of the Elementary Committee are: Fred J. Weaver, Chairman; Joyce Benbrook; Ann Peters; Irene Sauble; Esther Swenson; Henry Van Engen.

sense aspects of the problem.

One of the qualities of good mathematics is that it says precisely that which must be said. It follows that when one does say precisely what is meant, it is more easily understood. This is one of the common-sense rules of good communication, and it is a common-sense rule of good teaching. If the teacher says precisely what he means, and if he has given the pupil the proper background of experience, then the pupil should learn more readily. An example may help clarify the thought we are trying to express.

There are many things about the mathematics of the junior high school that need careful thought, but none deserve more thought and attention than the introduction to the use of symbols (usually letters of the alphabet) to hold open a place for the name of an arbitrary member of a set of numbers in a mathematical sentence. This introduction usually takes place when studying areas in the sixth or seventh grade. While engaging in such work, this generalization is developed: "Area equals Length times Width." The next step in this all too common sequence informs the pupil that the words "Area," "Length," and "Width" may be abbreviated to produce the formula $A = LW$.

This procedure illustrates how a teacher can fail to say what is really involved when developing an important mathematical idea. The pupil gets the idea that algebra is some kind of shorthand notation, and, indeed, it is often called just that when teaching the above formula. This teaching procedure gives the pupil a false idea about how letters of the alphabet are used in mathematics. With such a false start, is it any wonder that upon occasion mathematics is hard to learn? Pupils encounter many difficulties when making the abstractions necessary for understanding algebra; it is not necessary to add to their task by leaving the fundamental ideas unsaid. This situation can only be remedied by postponing the introduction of formulas until the pupil has grasped, in part, the fundamental idea of how variables are used in mathematics. This can be done, and it should be done, in the seventh grade.

There are other characteristics of good mathematics that can be used to illustrate how learning can be facilitated. Here is just one more.

Mathematicians try to state the fundamental principles around which mathematics is organized. In actual practice, they try to make the number of these fundamental principles, usually called postulates, as small as possible. This is not always a feasible thing to do in the secondary schools. However, to make mathematics easier to learn it is important for the teacher of mathematics to have in mind these fundamental principles. They serve to organize the instructional sequence, and they serve to unify the subject for the pupil. The structuring of ideas is important for mathematical reasons, and it is important for psychological reasons.

In secondary work many instances can be found which illustrate how the failure to keep in mind the structure of mathematics has made mathematics harder to learn. To cite one such instance, let us turn our attention to the distributive law.

Too frequently, the distributive law is not even mentioned in secondary school classes. An understanding of this fundamental principle (postulate) and its uses would help to clarify mathematics for the pupil and prevent many mistakes so commonly seen at all levels of mathematical instruction. For example, failure to appreciate the proper use of the distributive law causes such errors as:

$$\frac{s+t}{s} = \frac{\not{s}+t}{\not{s}} = t.$$

But there is yet an even stronger point for condemning any program that does not give this postulate its proper place in mathematical instruction. It is frequently the practice to teach the multiplication of a polynomial by a monomial early in the ninth grade algebra course. Some weeks or

even months later, factoring is introduced. All of this is accomplished (one wonders how it could happen) without even mentioning that the distributive law is in operation in both instances. Furthermore, why should the distributive law be taught in two segments separated by weeks (maybe months) of instruction on other aspects of algebra? Factoring such expressions as $2x-6$ and finding the product of such expressions as $2(x+4)$ should be taught simultaneously if a real understanding of the distributive law is to be achieved. Such an organization would conform to the standards of good mathematics and it would make mathematics easier to learn.

Precise statements and adequate use of mathematical postulates for instructional purposes are only two of the many ways that could be used to characterize good mathematics teaching and, possibly, good mathematics. These, however, will serve to illustrate the main point. Good mathematics is easier to learn than bad mathematics. That being so, let's teach good mathematics.

What's new?

BOOKS

Secondary

Algebra and Its Use (Book II), Ethel L. Grove, Anne M. Mullikin, Ewart L. Grove, New York, American Book Company, 1956. Cloth, iv+538 pp., $3.40.

Intermediate Algebra (with answers), William G. Shute, William E. Kline, William W. Shirk, LeRoy M. Willson, New York, American Book Company, 1956. Cloth, ix+486 pp., $3.40.

College

An Introduction to Tensor Analysis, Leonard L. Barrett, Palo Alto, California, The National Press, 1956. Paper, 33 pp., $2.00.

Fundamental Concepts of Higher Algebra, A. Adrian Albert, Chicago, University of Chicago Press, 1956. Cloth, ix+165 pp., $6.50.

Lectures on the Icosahedron (2nd and rev. ed.), Felix Klein, translated by George C. Morrice, New York, Dover Publications, Inc., 1956. Paper, xvi+289 pp., $1.85.

Mathematical Analysis, E. J. Camp, Boston, D. C. Heath and Company, 1956. Cloth, xiv+610 pp., $6.00.

Miscellaneous

Fun With Figures, J. A. H. Hunter, New York, Oxford University Press, 1956. Cloth, xi+160 pp., $3.00.

BOOKLETS

Careers in Research Test and Development (Announcement No. 76-B), U. S. Civil Service Commission, Washington 25, D. C. 40-page illustrated booklet announcing engineering and physical science positions in the Potomac River Naval Command, free.

The Electronic Brain and What It Can Do, Science Research Associates, 57 West Grand Avenue, Chicago 10, Illinois. 64-page illustrated booklet written by Saul Gorn and Wallace Manheimer, 60¢ each (discount on quantity orders).

Life Insurance Fact Book, 1956, Institute of Life Insurance, 488 Madison Avenue, New York 22, New York. 112-page booklet with many charts giving basic factual information about the life insurance business, free.

Models for Teaching, Visual Instruction Bureau, Division of Extension, The University of Texas, Austin 12, Texas. 40-page illustrated booklet explaining models—their use and construction—prepared by Martha F. Meeks, $1.00 each, postpaid.

Study on the Use of Science Counselors, John R. Mayor, Science Teaching Improvement Program, 1515 Massachusetts Avenue, N.W., Washington 5, D. C. 16-page booklet, Free while supply lasts.

MISCELLANEOUS MATERIALS

Paper Cut-Out Puzzles, Cletus Oakley, Haverford College, Haverford, Pennsylvania. Two reprints from *Jack and Jill* showing geometric puzzles, free (desired quantity for a school should be requested by principal or mathematics department).

Tax Teaching Kits, Public Information Division, Internal Revenue Service, Washington 25, D. C. Kit for general or farm course in preparation of income tax return; includes guides and handbooks, blank tax forms, and enlarged copies of the blank return form; material applies to the 1956 tax year; free.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKLETS

Aids and Shortcuts in Arithmetic, Ernst F. Bass, Portland, Oregon, Ernst F. Bass, 1956. Paper, 38 pp. (mimeographed), $1.00.

It is the author's purpose in his booklet "... to point out that our adopted counting system, the DECIMAL SYSTEM, has many advantages, and to show how these can be utilized to make addition, subtraction, multiplication and division easier and faster than we had thought possible."

The booklet is divided into fourteen sections or foci. The scope of content covers things such as: complementary method of subtraction; multiplication by 11, by 5 and multiples of 5, by 9 and multiples of 9; uses of common aliquot parts of 1; multiplication utilizing the factors of the difference of two squares; old or discarded methods of multiplication; tests of divisibility, including applications of casting out 9's; magic squares; and aspects of the metric system. The booklet concludes with a brief annotated bibliography of related books and booklets.

The nature of the content, coupled with the manner in which it is written, makes the booklet most appropriate for use at the senior high school level, with some sections appropriate for use with junior high school pupils.

The author's approach to the various shortcuts is not consistent. Some shortcuts are presented solely on the basis of procedure to be followed. Others are presented in a manner which gives an indication of their rational basis. Some shortcuts can be learned and applied quite mechanically. Others require some facility with various number relations if they are to be used effectively.

It is unfortunate that some loose or questionable statements and expressions appear in the booklet. "Since multiplying by 10, in the decimal system, is simply done by adding a zero to the number multiplied . . ." (p. 11). This may be true when multiplying an integer by 10, but not when multiplying a decimal fraction by 10. On page 14 we find extensive cancellations shown in which the 1's resulting from the various divisions are not shown. On the same page we see evidence of the false analogy that we cannot add halves and thirds for the same reason we cannot add houses and cars, or pears and peaches.

In general it may be said that parts of this booklet may prove to be of interest and value to selected students or groups of students. Teachers must exercise care and discrimination, however, in their use of this material in the mathematics classroom.—*J. Fred Weaver, Boston University, Boston, Massachusetts.*

BOOKS

Calculus, William K. Morrill, Princeton, D. Van Nostrand Co., Inc., 1956. Cloth, x+537 pp., $6.00.

The author of this textbook begins with a brief treatment of numbers, defining a real number intuitively as the number corresponding to the point common to a nested sequence of rational intervals approaching zero in length. This enables him to give arithmetical proofs for several theorems which are usually stated without proof in such texts. Although there is no pretense of absolute rigor, these proofs will probably be more satisfying to the careful student than the usual appeal to geometric intuition. The innovation which appeals most to the reviewer is the early definition of "null function" and "null sequence" and the consistent use of these concepts thereafter in proofs. It is doubtful that the unusual definitions of differentiability and of derivative will make these concepts as clear to the student as the traditional definitions which make these notions depend more explicitly on the concept of limit, even though, as the author states in his preface, his definitions are more easily generalized.

The author takes great pains to give a full and careful treatment of each topic, yet he has allowed many inconsistencies to remain which should have been removed in a careful reading of the manuscript. In the definition of instantaneous speed, page 44, it is specified that the limit is to be taken when the independent variable is zero, in spite of the careful discussion on page 29. In the discussion leading up to the definition of definite integral, continuity of the integrand is assumed, but when the definition is given, only boundedness is assumed. On page 376, "corresponding positive series" is used, but the reviewer can find no definition of this term. It would have been almost as easy to say "series of absolute values of the terms of the given series." The use of the word "hence" on the last line of page 29 makes it seem that the converse of the theorem is a consequence of the theorem, whereas both are consequences of the definition given two lines above. There are

many more such inconsistencies. It is hoped that these will all be removed at the next printing.

The reviewer believes that the "delta" notation for increment is not as suggestive to the student as the explicit formula. On the other hand, he thinks that the author's use of the notation $y=y(x)$ instead of the usual $y=f(x)$ is more suggestive. Many students at the elementary calculus level are not accustomed to the use of mathematical induction, and where this method is used (e.g., p. 65) it should be called by name.

The book is well made, is relatively free from typographical errors, and is printed in a form which makes it easily readable. In common with practically all textbooks, it lacks a sufficiently exhaustive index. It has excellent theoretical and applied problems, but could be improved by the addition of more drill problems in the technique of differentiation. The author is so careful to treat each topic fully, including all exceptional cases, and has incorporated so many other excellent features in the book that it is hoped that there will soon be a new edition with the mentioned defects remedied. Although the reviewer has not used it in class, he believes that, except for the criticisms above, the text would be very teachable. From the preface it appears that the material has been used for some time at the Johns Hopkins University and has proved successful as a classroom text.—*L. E. Bush, Kent State University, Kent, Ohio.*

From Zero to Infinity, Constance Reid, New York, Thomas Y. Crowell Company, 1955. Cloth, 145 pp., $3.00.

A former teacher, Mrs. Reid, now housewife and free-lance writer, has found numbers fascinating. In writing *From Zero to Infinity* she hoped that others might likewise come to find natural numbers fascinating. Through the unique personality of each of the digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9—the author exploits ideas of number theory. In order, these are developed as positional notation, the infinite set of natural numbers (including odd and even numbers), the binary systems and other number bases, primes and composites, squares, perfect numbers, and the use of Standards' Western Automatic Computer (SWAC) in computing the thirteenth perfect number, figural numbers, unique numbers, cubical numbers, and conditions of congruence. The reader will find tests and puzzles intriguing. Although no one reader may be interested in all of them, some pages are certain to challenge each reader. Biographical sketches, as a part of the exposition, acquaint the reader with such mathematicians as Gauss, Fermat, and Euler. Documentation of facts does not detract from the pleasure of reading this book.

Extending thought beyond the printed page, the reader is able to gather the emphasis given to zero as a place holder and number and just as a place holder. But the author's use of the words *invention* and *discovery* is confusing. The reader may feel that the author contradicts herself. Overemphasis on the interest of the Greeks in logic suggests to the reader that this was the primary concern of the Greeks. Historical records parallel the two-fold Greek interest in number—"logistic" (computation) and "arithmetica" (logic).

Despite the obscurities in some of these statements, the book certainly deserves recommendation to secondary and college students as well as to adults interested in number. This book undoubtedly would motivate many to more serious thinking in number theory. Certainly, in a delightful way, it well serves the purpose of the author—to get others interested in the fascinating theory of numbers.—*E. Glenadine Gibb, Iowa State Teachers College, Cedar Falls, Iowa.*

Mathematical Analysis, Walter T. Hamilton and John Raymond Hamilton, New York, Harper and Brothers, 1956. Cloth, xiii + 379 pp., $4.50.

This text was written for a first-year college mathematics course. The authors have attempted to satisfy the needs of pre-calculus students as well as those who will probably study mathematics but one year. Although it has the subtitle "A Modern Approach," the book is modern only in spots. Still, there is enough departure from the algebra-trigonometry-analytic geometry tradition to keep this text out of the conservative class.

The range of subject matter is wide, assuming acquaintance with high school algebra and geometry, proceeding through differentiation and integration. The calculus sections are not treated in detail.

The authors have attempted to show examples of rigorous mathematical proof, while also using informal and intuitive approaches in other instances. This is a difficult task and is not really accomplished to any great degree in this text. One might reasonably expect, from reading title and preface, a presentation of elementary college mathematics that "hangs together." This does not. The authors used admirable restraint in selecting and setting forth some principles of logic and classes in Chapter 1, but made very little specific use of them in subsequent chapters. In presenting sequences, and also functions, careful and correct definitions of the limit process are given. This is good, but applying the definition of limit, either rigorously or intuitively, is not done at all. Thus this topic yields little real understanding of the concept to the student.

An attempt is made to study the logic of solving linear equations but there is, if not an error, a very misleading statement on page 90: "To solve an equation we must write a new equation in the desired form, which is a *logical consequence* of the original." The question of

validity should be tied up with proving the converse of this "logical consequence," but it is not.

In Chapter 2 is found a device, new to textbooks, for showing intuitively how real numbers are multiplied. Its geometric nature should make it easily grasped, and it does help show why we say the product of two negatives is a positive. Also very helpful is the statement (p. 46), ". . . there is no such quantity as infinity (∞) within the system of real numbers."

In the circular functions chapter, the authors have done well in following their definition of an angle as a real number. This lays a good basis for understanding the function aspect of trigonometry.

The presentation of loci in Chapter 11 is confused. The definition of locus is made in terms of a curve "representing" an equation, yet it is really difficult to find what "represent" means. Then, in an attempt to prove rigorously that the locus of $Ax+By+C=0$ is a straight line, the authors fail to describe *what* straight line they will prove to be the locus. Throughout this proof, this line is referred to as "the locus" before it is known to be. It is really difficult to judge what has been done in this proof.

To conclude, this reviewer would find it difficult to teach this text, in either of the student groups for which it is intended.—*Lyman C. Peck, Ohio Wesleyan University, Delaware, Ohio.*

Plane Geometry, Rolland R. Smith and James F. Ulrich, Yonkers, New York, World Book Company, 1956. Cloth, viii+536 pages, $3.84.

Plane Geometry by Smith and Ulrich is close enough to a traditional treatment of geometry to satisfy most conservative teachers, and yet has enough unusual features to make it appeal to forward-looking teachers.

One rather striking feature is the extent to which the authors try to help the student see the meaning of what he is doing. Not only does each chapter start with some comments on what is ahead, but many pages start with a brief, simple statement of what the student is about to do and why. The discussion speaks directly to the student. The early part of the book leads gradually by careful steps from finding conclusions by measurement, through inductive reasoning, to if-then reasoning and demonstrative proof. Frequently there is a page "Before you study Theorem . . . " to help the student think through an analysis of the method of proof before he looks at the new theorem. Suggestions and practice are given in recognizing whether a proof is a good one. A few incorrect attempts at proofs are given to furnish practice in detecting errors. Specific spots where the student might have trouble (such as recognizing alternate interior and corresponding angles) are recognized, and special explanation and practice are given.

In the chapter on measurement of angles, the authors depart from the traditional treatment and use algebra almost entirely for the proofs. For teachers who may not care to use the algebraic proofs, they offer an alternative approach by studying particular angles and arcs and arguing informally that the steps and relationships would be the same no matter what angles or arcs are chosen. In line with the greater use of algebra, there is a section in the back of the book called "Maintaining Basic Meanings and Skills" which gives practice in the algebra needed.

Teachers will notice that quite a number of theorems which are traditionally proved are postulated in this book. Some may think there are too many postulates, but in general this trend is good because it allows a more gradual increase of difficulty in the theorems proved, and it allows more time for some of the aspects other than proving theorems. Some pages late in the book show nicely how a statement which was at one time postulated may later be proved. Possibly it would have been better if the authors had not marked these pages "optional."

The allowance for adjustment to individual differences is considerable. On the one hand, a minimum course is clearly indicated. More of the theorems may be postulated and only the basic exercises need be studied. On the other hand, much opportunity is given for the more able students. There are extra exercises and pages called "Problems for Pacemakers." Particularly noteworthy are the "optional" pages which occur frequently—every ten or twelve pages. These treat topics such as practical uses of isosceles triangles, trisecting angles, multiconverse statements, various phases of solid geometry, non-Euclidean geometry, necessary and sufficient conditions, compound curves, navigation, the cycloid, a linkage, and precision in measurement. While the treatments on these pages are necessarily brief, they are stimulating. In addition to the optional pages, there are three optional chapters on geometric constructions, analytic geometry, and logic respectively.

There are a few weak spots. The concepts of distance and length do not seem to be clarified beyond saying (p. 112) that "The distance between two points is the length of the straight line between them." The distance between parallel lines is defined (p. 140) long before it is proved (p. 175) that there can be such a distance. On page 362 the substitution axiom seems to be used as justification for replacing a triangle with another triangle congruent to it. Some teachers may not be satisfied with the informal treatment of the measurement of the circle. All of these criticisms may be a matter of how formal or how informal we should be with tenth graders.

In format the book is excellent. The drawings are clear, the pages are easy to read, color is used to good advantage, and the cover is attractive.—*Henry Swain, New Trier Township High School, Winnetka, Illinois.*

INSTRUCTIONAL MATERIALS

Basic Algebra Data-Guide, and *Slide Rule Data-Guide*, Student Marketing Institute, 375 Fifth Avenue, New York 16, New York. Price for either guide, 69¢ each postpaid, or 42¢ each plus postage in lots of twelve or more.

Each of these guides is a white plastic sheet $8\frac{1}{2}'' \times 11''$ with holes punched for a 2-ring or a 3-ring notebook. The basic algebra guide is written by Myron F. Rosskopf, and the slide rule guide by Mario G. Salvadori and Jerome H. Weiner. The sheets are printed on both sides in three colors—black, red, and blue—with the text arranged in three columns. Each guide presents all the rules or laws pertinent to the title in a concise and compact form. The general law is stated in words and again in symbols, followed by an example.

Both guides are attractive, durable, and printed clearly. Their value stems mainly from the summarizing effect of presenting a mass of mechanical rules in an area equivalent to the two sides of a sheet of notebook paper.—*Richard D. Crumley.*

Optical Illusions, J. Weston Walch, Publisher, P. O. Box 1075, Portland, Maine. 24 charts, $1.00 for set.

This set of twenty-four charts, each printed on cardboard $8\frac{1}{2}'' \times 11''$, are blown-up reproductions from the book *Math Can Be Fun* by Louis Grant Brandes. Each chart has an optical illusion presented on one side with a brief discussion of the illusion on the back. An introductory statement for the set is included.

The illusions are drawn well and will be stimulating to most high school students.—*Richard D. Crumley.*

In 1954 nearly 345,000 undergraduate students received Federal financial support. Of these, about 82,400 were enrolled for "major" purposes in the sciences—about 46,000 in engineering. Educational costs (to the government) averaged about $1,000 per year for these undergraduate students. Ninety-seven per cent of this aid came via the veterans educational benefits provision of various veteran readjustment legislation. Another two per cent came from N.R.O.T.C. scholarships. Thus, the report points out, undergraduate eligibility for Federal support came about almost wholly as a result of military service, performed or obligated. —*Taken from* Newsletter No. 99, *Engineering Manpower Commission of Engineers Joint Council and Scientific Manpower Commission.*

The percentage of pupils in elementary algebra and ninth grade general mathematics is lower in the New England and Middle Atlantic regions than in the other regions. These two regions, however, are among the top in the percentage of pupils in plane geometry, trigonometry, and intermediate algebra.—*Pamphlet No. 118, U.S. Department of Health, Education, and Welfare.*

• TESTING TIME

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

Mathematics testing in Russia

What answers do you get to the following algebra problems?

Example 1.

(a) Determine the exponent n (n is a positive whole number) in the expansion of

$$\left(\frac{1}{5}x+\frac{2}{5}\right)^n$$

by diminishing powers of x, if the tenth member of this expansion (counting from the beginning) has the largest coefficient.

(b) Solve the system of equations:

$$7\sqrt[3]{xy}-3\sqrt{xy}=4$$
$$x+y=20$$

(c) Let x_1 and x_2 be the roots of equation:

$$x^2-ax+a-1=0$$

where a is a real number. Find the value of a for which the magnitude of expression $x_1^2+x_2^2$ will be least.

Example 2

(a) Two trains proceeding in opposite directions, one an ordinary train going from A to B and the other an express train from B to A, were delayed at an intermediate station C for two hours. By increasing the speed of their trains by 25 per cent, the engineers brought the trains, respectively, to B—48 minutes late; and to A—1 hour 36 minutes late.

Determine the initial speed of trains if the speed of the express train was 20 kilometers per hour greater than the speed of the ordinary train, and the distance from A to B is 360 kilometers.

(b) Solve the system of equations:

$$\frac{1}{x}-\frac{1}{y}=\frac{1}{36}$$
$$xy^2-x^2y=324$$

(c) Determine three numbers forming a geometric progression if their sum is equal to 21 and the sum of their reciprocals is equal to 7/12.

(d) Solve the equation:

$$lg\sqrt{1+x}+3lg\sqrt{1-x}=lg\sqrt{1-x^2}+2$$

Example 3.

(a) Two sums of money, a total of 50,000 rubles, were placed in a savings bank at 3 per cent annual interest. The first sum remained on deposit 4 months longer than the second sum, but each yielded 600 rubles interest.

Find how large each sum was and for how long each was deposited. It is known that neither sum remained on deposit for longer than one year.

(b) Solve this system of equations:

$$7x-11y=\sqrt[3]{x+y}=\sqrt[5]{x+9y}$$

(c) Prove that if the coefficients of equations:

$$x^2+px+q=0$$

and

$$x^2+p_1x+q_1=0,$$

where p, q, p_1, and q_1 are real numbers, are

related so that

$$pp_1=2(q+q_1),$$

then at least one of the equations has real roots.

(d) Find the number of dissimilar members among the members of expansion:

$$(a+b+c)^n$$

Example 4

(a) Water can flow into a tank through four pipes. If pipes 1, 2, and 3 are open, the tank will be filled in a hours; if pipes 2, 3, and 4 are open, in b hours; if pipes 3, 4 and 1, in c hours; and if pipes number 4, 2, and 1 are open, the tank will be filled in d hours. Determine the time needed to fill the tank if each of the pipes alone is open.

(b) Solve this equation graphically:

$$\sqrt[4]{x}=x+0.2$$

(It is required to determine the number of real roots and to determine them with an accuracy to the nearest .1).

(c) How many different ten-cipher numbers can be written using only the ciphers 1, 2, and 3 with an added limitation that cipher 3 be used only two times?

(d) Determine the number of coefficients different from zero in the expansion:

$$(1+x^2+x^5)^{20}=a_0+a_1x+a_2x^2+\cdots+a_{100}x^{100}$$

If you were able to solve correctly at least half of these, then you could probably be admitted as a freshman at the Moscow State University to study in the Mechanics-Mathematics Department! The questions above are actual sets of algebra questions from the 1950–52 written entrance examinations to that university.

For the past several years, the Center for International Studies of the Massachusetts Institute of Technology has been studying the educational system of the USSR. Of particular interest to the MIT project have been Russian efforts to train engineers, mathematicians, and scientists. Statesmen and engineering experts in this country claim that these efforts have resulted in the Russians outstripping the U.S. in numbers of such personnel graduated each year.

Of greater importance and interest than this concern about numbers of people graduated is the question of the quality of Russian education in the sciences and applied fields. In the course of patient and objective investigation of this question, the MIT group came into possession of examination questions in mathematics given to applicants for the University of Moscow.

To draw any definite conclusions about the quality of Russian education, we would need more information than these sample examination questions—information about administration, scoring, and use of the exams, as well as information about what occurs in the classroom and in the minds of teachers. However, it is interesting and perhaps of some value to speculate about such things on the basis of what we can find in these problems. For example, on the basis of this fairly large sample, what can we guess about the Russian precollegiate curriculum in mathematics?

Speaking as the editors, we found ourselves engrossed in working through these problems and in trying to analyze what skills and understandings were needed for success. Rather than give our own analysis here at this time, we hopefully suggest that you may enjoy working through these problems yourselves and jumping to your own conclusions—you may also want to try some of them on your students. In the next issue, we shall give our answers and our analyses for your comparison.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Thirty-Fifth Annual Convention

Bellevue-Stratford Hotel, Philadelphia, Pennsylvania
March 27–30, 1957

Twentieth-century mathematics, and its implication for teaching mathematics from grade one through the junior college, will be highlighted at the Thirty-fifth Annual Convention of the National Council of Teachers of Mathematics at Philadelphia. The current thinking and significant experimental and research study in modernizing our mathematics curriculum will be presented by outstanding leaders in mathematics and education.

Included among the special events is the opening address on Thursday evening by Dr. Mario Salvadori, Professor of Civil Engineering at Columbia University, entitled "The Universal Fear of Mathematics." At the general session on Friday morning the Twenty-third Yearbook of the Council will be presented. This book, written by thirteen mathematicians of the highest rank, is entitled *Insights into Modern Mathematics.* This book will be outlined and reviewed by Dr. Bruce Meserve of Montclair, New Jersey.

At the banquet on Friday night, an outstanding woman in the field of mathematics, Dr. Marguerite Lehr, of Bryn Mawr College, will deliver the address with the intriguing title "More Knowledge or More Knowing." The luncheon speaker on Saturday is the well-known, popular Colonel Robert C. Yates of Virginia's William and Mary College, who will speak on "The Student You Send to College."

On Thursday afternoon, Dr. Phillip Jones of Ann Arbor will conduct a student hour for six hundred high school juniors and seniors outstanding in their work in school mathematics. His title is "Mathematical Wonders." Our Saturday morning session we are keeping as a surprise package which we hope will be "A Delight and Prelude" to all the sessions the rest of that day.

There will be book exhibits, commercial exhibits, student exhibits, and a trip to the famous Benjamin Franklin Institute of Science. The social events promise a gala time for accompanying wives, husbands, and friends whose interests are other than mathematics. The various section meetings from Thursday morning to late Saturday afternoon feature the men and women who have significant and practical contributions to bring to our profession.

Every elementary school, secondary school, and college should have at least one representative at this convention.

Program

Wednesday, March 27

Elementary and secondary schools in Philadelphia having special programs or doing outstanding work in mathematics will be open for visits this entire day. In the evening, a visit to the Franklin Institute of Science and the Planetarium offers both entertainment and education. Come

to the convention early and enjoy these events.

Thursday, March 28: Morning

The Annual meeting of the Delegate Assembly convenes at 9:00 A.M. Visitors as well as official delegates to the convention are welcome to this three-hour session. For those not attending the Assembly there are two sightseeing trips, one to historic Philadelphia, the other to Valley Forge, a showing of mathematical films, and also further opportunity to visit Philadelphia schools. Time has also been provided for meetings of the several committees of the Council.

Thursday: Afternoon and Evening

From 2:00 to 3:45 P.M. the Convention will get into real swing with six outstanding sectional meetings: one at each of the levels, elementary, junior high school, senior high school, and college; another on Research in Mathematical Education; and a special section, sponsored by the Association of Mathematics Teachers of New Jersey on "Space Perception." This section features Lt. Louis J. Cavello, USMC Artillery School, Quantico, Virginia.

From 3:45 to 5:15 P.M. there are laboratory sections on the elementary, junior high school, and senior high school levels, and from 4:00 to 5:00 P.M. the High School Youth Forum.

The State Representatives of the NCTM will meet with Mr. Ahrendt, the Executive Secretary of NCTM, and Miss Mary Rogers, chairman of the Membership Committee, from 7:00 to 7:45 P.M.

At the General Session beginning at 8:00 P.M., Dr. Allen H. Wetter, Superintendent of Schools of Philadelphia, will give an official welcome to be followed by Dr. Salvadori's address. The evening concludes with a reception to all convention members given by the host organization, the Association of Teachers of Mathematics of Philadelphia and Vicinity.

Friday, March 29: Morning

This is really modern mathematics day. After the opening session in which the YEARBOOK will be presented, there will be four large section meetings at each of the several grade levels. The elementary school section will feature Maurice Hartung of Chicago and Agnes Gunderson of Wyoming. The junior high school session has John Mayor and Alice Hach considering the mathematics improvement at this level. The senior high school session will feature "Modern Concepts of Sets, Sentences and Relations, and Their Implication for the Secondary School Program." The remaining section will consider "Mathematics in General Education," especially the "Role of Statistics and Mathematics in Everyday Life."

Friday: Afternoon and Evening

The afternoon session from 2:00 to 3:45 P.M. breaks away from grade levels (horizontal sections) and provides vertical sections on the topics Evaluation and Testing, International Relations, the Continuous Curriculum, the Training of Teachers, and Enrichment, Clubs and Contests. There will also be a reshowing of mathematical films of interest from 2:00 to 4:00 P.M.

The feature of the afternoon will be the Annual Business Meeting of the Council. This is an opportunity for every member to hear what the National Council is doing in Mathematics Education. The work of your Board of Directors, the work of the committees, the election results, the publications, and proposed projects of the Council will be presented. Members are free to raise questions, make proposals, and direct the thinking of the Board to matters of concern to the Council at large. We hope all members will plan to attend this meeting.

The day's activities conclude with the annual banquet, always an outstanding and rewarding affair and with a real treat in store in the address of Dr. Marguerite Lehr.

Saturday, March 30: Morning

The opening session is planned for 9:00 A.M. with a mathematician as speaker who has gained world renown for his popularization of mathematics. This address was planned for this session so that the many teachers who could not attend the meetings on Thursday and Friday would have the opportunity to hear this speaker. Following his address, there will be four sections, one at each level of instruction, and a fifth section, sponsored by the teachers of Philadelphia on "Planning the Mathematics for a Large City System." In these sessions, at your level of interest, you will hear of the leading movements in mathematical education both in our country and abroad.

The luncheon promises to be a gala affair with Mr. Linton of Philadelphia toastmastering and Colonel Yates lecturing in a fashion that is all his own.

Saturday: Afternoon

The convention does not end until 3:45 P.M. From 2:15 to 3:45 we have arranged the most important sections of the convention. Each of these sections deals with "The *Teaching* of Mathematics" with concrete practical suggestions on content and methods. For grades 1 to 8, there is a section on "Specific Techniques for Improving Instruction." For the secondary school teachers there are two sections, one on specifics for the abler students, and one on specifics for the noncollege-bound students. At the college level, there is a section on specific content for the freshman year. All members should plan to attend one of these sections.

Notice is hereby given, as required by the Bylaws, that the Annual Business Meeting of the National Council of Teachers of Mathematics will be held during the Thirty-Fifth Annual Meeting, March 28–30, 1957, at the Bellevue-Stratford Hotel, Philadelphia, Pennsylvania.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING

March 28–30, 1957
Bellevue-Stratford Hotel, Philadelphia, Pennsylvania
M. Albert Linton, William Penn Charter School, Philadelphia, Pennsylvania

JOINT MEETING WITH NEA AND NSTA

July 1, 1957
Philadelphia, Pennsylvania
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

SUMMER MEETING

August 19–21, 1957
Carleton College, Northfield, Minnesota
Margaret Linster, St. Louis Park High School, Minneapolis 16, Minnesota, or Kenneth O. May, Carleton College, Northfield, Minnesota

Other professional dates

Women's Mathematics Club of Chicago and Vicinity
February 9, 1957
Place to be announced.
Ruth Woerner, 11715 S. 82nd Court, Palos Park, Illinois

Chicago Elementary Teachers Mathematics Club
March 11, 1957
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Genevieve E. Johnson, Volta School, Chicago, Illinois

Joint Meeting of Women's Mathematics Club of Chicago and Vicinity with Men's Mathematics Club of Chicago and Vicinity
March 15, 1957
Tearoom of The Fair Store, State and Adams Streets, Chicago, Illinois
Ruth Woerner, 11715 S. 82nd Court, Palos Park, Illinois

Annual Meeting of Georgia Mathematics Council
March 22, 1957
Georgia State College of Business Administration, Atlanta, Georgia
Sara Durham, Coffee County High School, Douglas, Georgia

Tenth Annual Conference on Elementary and Secondary Mathematics
April 6, 1957
Illinois State Normal University, Normal, Illinois
Douglas R. Bey, Illinois State Normal University, Normal, Illinois

Sectional Meetings of the Illinois Council of Teachers of Mathematics
March 23, 1957, Granite City, Illinois
March 30, 1957, Joliet, Illinois
March 30, 1957, Carbondale, Illinois
April 10, 1957, Charleston, Illinois
April 13, 1957, Kewanee, Illinois
Francis R. Brown, Illinois State Normal University, Normal, Illinois

The following meetings are institutes for teachers of mathematics sponsored by the National Science Foundation. The dates given are approximate. The person named in each case is the director of the institute. Unless stated otherwise, the mailing address of each director is at the institution given above.

June 24–July 20, 1957 (For college teachers only)
University of Colorado, Boulder, Colorado
R. E. Langer, University of Wisconsin, Madison, Wisconsin

June 10–August 3, 1957
University of Kansas, Lawrence, Kansas
G. B. Price, Department of Mathematics

July 8–August 16, 1957
Columbia University Teachers College, New York 27, New York
H. F. Fehr, Department of Mathematics

June 17–August 9, 1957
Indiana University, Bloomington, Indiana
Marie S. Wilcox, Department of Mathematics

June 17–July 26, 1957
Miami University, Oxford, Ohio
T. C. Holyoke, Department of Mathematics

June 22–August 23, 1957
Montana State College, Bozeman, Montana
J. W. Hurst, Department of Mathematics

June 1–July 27, 1957
Polytechnic Institute of Puerto Rico, San German, P. R.
Mariano Garcia, College of A & M Arts, University of Puerto Rico, Mayaguez, P. R.

July 1–August 9, 1957 (For junior high school teachers)
State Teachers College, Oneonta, New York
Vera Sanford, Department of Mathematics

June 17–July 27, 1957
State University of Iowa, Iowa City, Iowa
L. A. Knowler, Department of Mathematics

July 8–August 2, 1957
University of Buffalo, Buffalo, New York
Harriet F. Montague, Department of Mathematics

June 24–August 2, 1957
University of Chicago, Chicago, Illinois
A. L. Putnam, 411 Eckhart Hall

June 17–July 27, 1957
University of Colorado, Boulder, Colorado
B. W. Jones, Department of Mathematics

July 1–August 16, 1957
University of Massachusetts, Amherst, Massachusetts
R. W. Wagner, Department of Mathematics

June 21–August 6, 1957
University of Notre Dame, Notre Dame, Indiana
A. E. Ross, Department of Mathematics

June 17–August 9, 1957
University of Wyoming, Laramie, Wyoming
W. N. Smith, Department of Mathematics

Academic Year, 1957–58
University of Chicago, Chicago, Illinois
E. P. Northrop, Mathematics Staff

Academic Year, 1957–58
University of Illinois, Urbana, Illinois
Joseph Landin, Department of Mathematics

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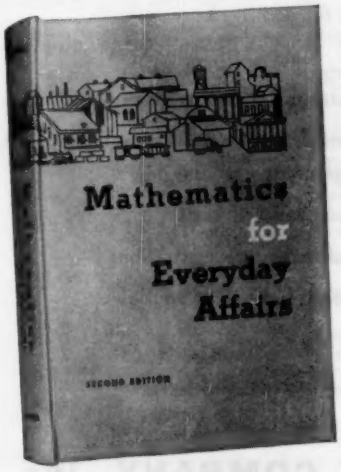

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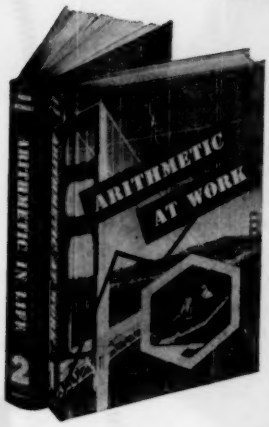

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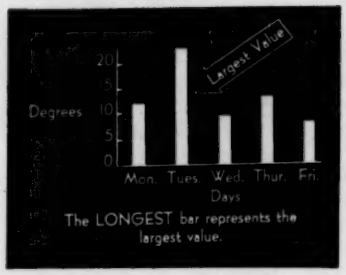

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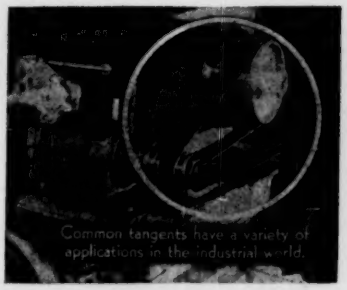

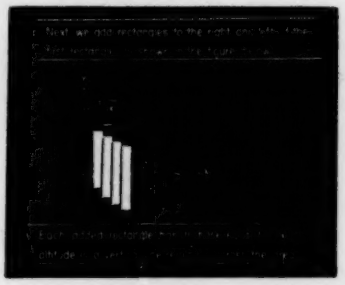

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